USING THE DIMENSION REDUCTION TECHNIQUE TO PROVE THAT CLIQUE TREES DEFINE FACETS FOR THE ASYMMETRIC TRAVELING SALESMAN POLYTOPE

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September 14, 1994

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ABSTRACT

In this paper the author develops a new general approach to proving that a given inequality is facet-defining. This approach is then used to show that clique tree inequalities define facets for the asymmetric traveling salesman problem.

Using the dimension reduction technique to prove that clique trees define facets for the asymmetric traveling salesman polytope

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1 Introduction

Given a complete graph G with costs on the edges, the traveling salesman problem (TSP) consists in finding the minimum cost hamilton cycle in G. The asymmetric traveling salesman problem (ATSP) consists in finding the minimum cost hamilton cycle in a complete directed graph G, where each arc has a cost.

Suppose a subgraph of G induced by a set U of vertices is a complete subgraph. In that case we call this subgraph a clique of G. A clique tree in G is a set of cliques of vertices having the following structure. Some cliques in the clique tree are called *handles* and others *teeth.* They have the
following properties:

(i)The handle cliques are pairwise disjoint.

(ii) The teeth cliques are pairwise disjoint.

(iii) The number of teeth that a handle intersects is an odd number that is at least three.

(iv) Every tooth has at least one node which is not in any handle.

(v)The graph of handles and teeth is connected.

(vi) Removing any non-empty intersection of a handle and a tooth disconnects the graph.

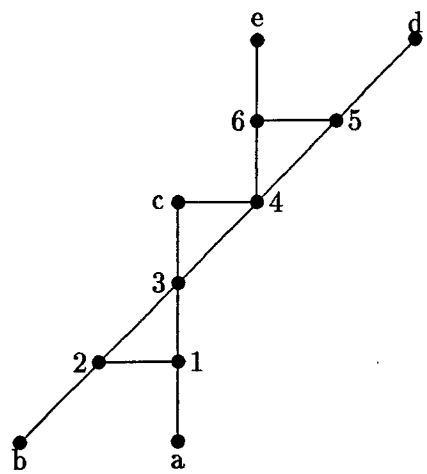


Figure 1: a simple clique tree

Note that a clique tree might not include all the vertices of the underlying graph.

A clique tree is *simple* if the intersection of any handle with any tooth is at most one. Figure 1 gives a drawing of a simple clique tree.

The handles for this clique tree are $\{1,2,3\}$ and $\{4,5,6\}$. The teeth are $\{a,1\},\{b,2\},\{d,5\},\{e,6\}$, and $\{c,3,4\}$.

Let H_i for $i \in \{1..r\}$ be the handles and T_j for $j \in \{1..s\}$ be the teeth of a clique tree. Let t_j be the number of handles that tooth T_j intersects. A valid inequality corresponding to this clique tree is:

$$\sum_{i\in\{1..r\}} x(E(H_i)) + \sum_{j\in\{1..s\}} x(E(T_j)) \leq \sum_{i\in\{1..r\}} |H_i| + \sum_{j\in\{1..s\}} (|T_j| - t_j) - (s+1)/2 \quad (1)$$

[1]. If this clique tree is simple, then the coefficient for every arc variable in (1) is either 1 or 0 depending on whether the arc is or isn't in the clique tree. In the former case the arc is said to *count* for the clique tree inequality. If $t_j = 1$ for tooth T_j then this tooth is defined to be a *free tooth.* It can be shown that in any clique tree having two or more handles, there are at least two handles having at least two free teeth each.

This paper establishes the following theorem:

THEOREM: All clique trees having two or more handles define facets of the asym-

metric traveling salesman polytope.

The theorem is given a constructive proof in this paper, using what the author calls the dimension reduction technique.

2 Background for the proof

2.1 The dimension reduction technique

Fundamental to the study of polyhedra are methods for the closely related problems of determining what the dimension of a polyhedron P is and whether a given valid inequality $ax \leq b$ is facet-defining for P. The principle technique for obtaining a lower bound k on the dimension of a polyhedron P is to find $k+1$ affinely independent points of P [2]. A valid inequality $ax \leq b$ is said to be facet-defining for P when with $P_1 := \{x \in P \,|\, ax = b\}$, one has $\dim(P_1) = \dim(P) - 1$. Thus the principle technique for showing that the above inequality is facet-defining is to determine first that $\dim(P_1) \leq \dim(P) - 1$,which is usually easy, and second to determine that $\dim(P) - 1$ is a lower bound on $\dim(P_1)$ as well. When the upper bound on the dimension of P_1 is shown to be $\dim(P) - 1$, the inequality $ax \leq b$ is said to define a proper face of P.

There two primary ways to show that $\dim(P_1) \geq \dim(P) - 1$, which would complete the proof that $ax \leq b$ defines a facet. The first way is to find $\dim(P)$ affinely independent points of P_1 [2] so that the above discussion applies. Of course, one needs to know the dimension of P first.

The other way involves the equality systems of P and P_1. The equality system of P, denoted by $P^=$, is the set of all equations which are valid on P. For a poyhedron defined on an n dimensional space, one has the relationship $\dim(P) = n - \text{rank}(P^=)$. So $ax \leq b$ defines a facet for P if and only if $\text{rank}(P_1{}^=) = \text{rank}(P^=) + 1$. Thus the second primary way to determine that a valid inequality $ax \leq b$ is facet-defining for P is to show that it defines a proper face of P and that any equation in $P_1{}^=$ is a linear

combination of equations in $P^=$ and the equation $ax = b$ [2].

In this paper another technique for obtaining a lower bound on the dimension of a polyhedron is developed. Since a lower bound is obtained, as described above, the technique of this paper can be used to determine that a given inequality defines a facet for a polyhedron of an integer program.

The basic ideas of this technique, which has been named the dimension reduction technique, are given for the first time here. Suppose you wish to determine the dimension of a polytope P. Choose an inequality $a_1x \leq b_1$ which is valid for P and whose associated hyperplane $a_1x = b_1$ intersects P. Define P_1 as:

$$P_1 = \{x \in P \,|\, a_1x = b_1\}$$

Hence P_1 is a face of P. Now if you're lucky, $a_1x = b_1$ can't be deduced from $P^=$, the equality system for P. In that case, $\dim(P_1) < \dim(P)$ since the rank of $P_1^=$ is at least one greater than the rank of $P^=$. On the other hand, if $a_1x = b_1$ can be deduced from $P^=$, then $P_1 = P$.

But our goal is to find a suitable inequality $a_1x \leq b_1$ so that $\dim(P_1) < \dim(P)$, which, by the above analysis, is equivalent to having $P_1 \neq P$. One way of insuring this is to find a point $x_1 \in P$ such that $x_1 \notin P_1$. Now note also that by assumption $P_1 \neq \emptyset$ and so P_1 is thus a non-empty proper face of P.

Now this process can be repeated on P_1. This involves choosing an inequality $a_2x \leq b_2$ which is valid and supporting for P_1 so that

$$P_2 = \{x \in P_1 \,|\, a_2x = b_2\}$$

defines a non-empty proper face of P_1. To insure that P_2 defines a non-empty proper face of P_1 it is sufficient to find a point $y_2 \in P_2$, (which shows $P_2 \neq \emptyset$), and a point $x_2 \in P_1$ so that $a_2x_2 < b_2$. If these points are found then $0 \leq \dim(P_2) < \dim(P_1)$ and hence $\dim(P_2) \leq \dim(P) - 2$.By iterating this procedure, eventually a polytope

P_k consisting of a single point will be found. Then we have

$$0 = \dim(P_k) \leq \dim(P) - k$$

$$\text{which implies} \qquad \dim(P) \geq k$$

It was mentioned above that to insure that P_i defines a non-empty proper face of P_{i-1}, which is step i in the dimension reduction procedure, it is sufficient to find two points, one each from P_{i-1} and P_i. However, in the dimension reduction procedure, finding the point $y_i \in P_i$ can be postponed until the next step, so at every step only a single equation and a single point which violates the equation have to be found.

So the dimension reduction technique for finding a lower bound for the dimension of P consists of the following procedure:

At stage $j+1$ do the following:

(i) If P_j consists of a single point, stop procedure with j being a lower bound on the dimension of P

(ii) Otherwise choose an inequality $a_{j+1}x \leq b_{j+1}$ which is valid and supporting for P_j and define P_{j+1} by

$$P_{j+1} = \{x \in P_j \,|\, a_{j+1}x = b_{j+1}\}$$

(iii) Find a point $x_{j+1} \in P_j$ such that $a_{j+1}x_{j+1} < b_{j+1}$. Increment j and proceed to the next stage.

The end result of this procedure is a nested sequence of polyhedra:

$$P \supsetneq P_1 \supsetneq P_2 ... \supsetneq P_k$$

where each polytope has a strictly smaller dimension than the one before it, and P_k consists of a single point.

Now let's explore when a lower bound k obtained from the dimension reduction procedure, is tight for P. Clearly this is when the dimension of the polytope goes down by exactly one at each stage of the dimension reduction procedure. But this condition is equivalent to

$$a_j x \leq b_j \text{ is a facet of } P_{j-1}$$

for all $j \in \{1, .., k\}$. So, if in fact one knew the dimension of P all along, one could be using this procedure to show that $a_1 x \leq b_1$ is a facet of P.

Finally, as demonstrated by the following theorem, when one uses the dimension reduction procedure to show that $a_1 x \leq b_1$ is a facet of P, the points found in step (iii) (from stage 2 onwards) form a list of $\dim(P) - 1$ affinely independent points in the face $\{x \in P \mid a_1 x = b_1\}$. Hence, presenting this list of points together with the unique point of the 0-dimensional polytope P_k which is the end result of the dimension reduction procedure, is an example of the first method for proving that $a_1 x \leq b_1$ is facet-defining for P.

THEOREM: The points obtained in step (iii) of the dimension reduction procedure form a set of affinely independent points.

PROOF: Denote these points by x_i for $i \in \{1, .., k\}$. Suppose that $\sum_{j \in J} \alpha_j x_j = 0$ and $\sum_{j \in J} \alpha_j = 0$. Choose j_0 to be the smallest $j \in J$ such that $\alpha_j \neq 0$. Then

$-\sum_{j \in J \setminus \{j_0\}} \frac{\alpha_j}{\alpha_{j_0}} x_j = x_{j_0}$.

Now, x_{j_0} doesn't satisfy the equation in step (ii) of stage j_0. However, by minimality of j_0, For all $j \in J \setminus \{j_0\}$, if $\alpha_j \neq 0$, then x_j satisfies the equation in step (ii) of stage j_0. But then $-\sum_{j \in J \setminus \{j_0\}} \frac{\alpha_j}{\alpha_{j_0}} x_j$ satisfies the equation in step (ii) of stage j_0, which is a contradiction. Hence, $\alpha_j = 0$ for all $j \in J$, which proves the theorem.

2.2 the form of the proof

The proof that all clique trees having two or more handles define facets for the asymmetric traveling salesman polytope consists of several parts, each of which proves a

particular proposition. These propositions are always of the form that if a certain class of clique trees define facets, then a slightly larger class of clique trees define facets as well. Combining these results together with the fact that on an underlying graph of at least 7 vertices, all the one-handle clique trees, which are called combs, define facets [3], one can then inductively show that all clique trees having two or more handles define facets. Recall that a simple clique tree is one where the intersection of any handle with any tooth is at most one. From here on, all clique trees are assumed to be simple clique trees since as is well known, if all simple clique trees define facets then all clique trees define facets due to cloning [4].

2.3 classes of clique trees used in proof

In addition to the concept of the free tooth are some other concepts. A *free handle* is defined as one which intersects at most one non-free tooth. A *small tooth* is defined to be a tooth containing only one vertex that isn't in any handle. A *small handle* is defined to be a handle intersecting only three teeth. The *middle of a handle* is defined to be the set of vertices of the handle which aren't in any tooth. Note that a handle doesn't necessarily have a non-empty middle.

Listed here are the classes of clique trees that are useful for the proof. These are to be refered to for the outline of the proof and the four parts of the proof.

C_1 = the class of one-handle clique trees (combs) for which the underlying graph has a vertex that isn't in the comb

C_k = the class of clique trees having more than one but k or fewer handles. for $k \geq 2$

$C_{=k}$ = the class of clique trees having exactly k handles.

T = the class of clique trees where every free tooth is small.

S = the class of clique trees that include a small free handle with no middle, which is called the *special handle.*

H_l = the class of clique trees for which there exists a free handle called the special handle with no middle which intersects exactly $2l+1$ teeth.

H = the class of clique trees for which there exists a free handle called the special handle with an empty middle.

M = the class of clique trees for which there exists a free handle called the special handle with a middle of cardinality at most one.

2.4 outline of proof

The first part of the proof shows that if all clique trees in C_k define facets, then all clique trees in $C_{=k+1} \cap T \cap S$ define facets as well.

The second part of the proof shows that if all clique trees in $C_k \cup (C_{=k+1} \cap T \cap H_l)$ define facets, then all clique trees in $C_{=k+1} \cap T \cap H_{l+1}$ define facets as well. Since $S = H_1$, the first two parts of the proof show that if all clique trees in C_k define facets then all clique trees in $C_{=k+1} \cap T \cap H$ define facets as well.

The third part of the proof shows that if all clique trees in $C_k \cup (C_{=k+1} \cap T \cap H)$ define facets, then all clique trees in $C_{=k+1} \cap T \cap M$ define facets as well.

The fourth part of the proof uses cloning to show that if all clique trees in $C_{=k+1} \cap T \cap M$ define facets, then all clique trees in $C_{=k+1}$ define facets as well.

Combining these four parts shows that if all clique trees in C_k define facets, then all clique trees in C_{k+1} define facets. It is already known that all clique trees in C_1 define facets since the underlying graph has a node that isn't in the comb. In spite of this stipulation, it follows that all clique trees in C_2 define facets, and hence it follows then that all (simple) clique trees define facets. Then, by cloning, it follows that all clique trees define facets.

Since whether a particular clique tree defines a facet may depend on the size of the underlying graph, each part of the proof establishes that the stated result holds for underlying graphs of any size large enough to contain the clique tree of two or more handles.

Part I
first part of proof

3 type of clique tree under investigation

Assume all clique trees in C_k define facets on any underlying graph, unless $k = 1$ for which we require that the underlying graph has at least one node which isn't in the comb. It has been shown by Fischetti [3] that all clique trees having one handle (combs) define facets, with one exception which is not in C_1. Let an underlying graph G be given. Let a clique tree in $C_{=k+1} \cap T \cap S$ on the graph G be given. Call this the *large clique tree.* The goal of the first part of the proof is to show that the large clique tree defines a facet. In labeling the vertices of this clique tree and those later on in this paper, numbers are used as labels for vertices which are in a handle and letters of the

english alphabet are used to label vertices which are in a tooth, but not in any handle. Look at the diagram of Fig. 1 on page #3 to visualize where the vertices of the large clique tree which are named below belong. Label the vertices of the special handle $1, 2$, and 3. Have 3 be the vertex that intersects the non-free tooth. Let the free teeth intersecting this handle be $\{a, 1\}$, and $\{b, 2\}$. Index the nodes in the tooth containing 3 which aren't in any handle over the index set I so that they are denoted as c_i for $i \in I$. Index the other nodes of this tooth except 3 over the index set $\overline{I}$ so that they are denoted as 4_i for $i \in \overline{I}$. Choose a special vertex c_0 from the family of vertices c_i for $i \in I$. Choose a handle other than the special handle that has at least two free teeth in this clique tree and denote the vertices of two of these teeth as $\{d, 5\}$ and $\{e, 6\}$, with 5 and 6 also being vertices in the handle. There always is such a handle, as was indicated on page #3.

3.1 clique trees used in first part

Call the given clique tree the large clique tree. Call the clique tree without vertices $1, 2, a$, and b on the underlying graph $V(BclG) = G\backslash\{1, 2, a, b\}$ the *big clique tree*, where $V(BclG)$ is read "vertices of the big clique tree graph". Call the clique tree without vertices $1, 2, a, b$, and 3 on the underlying graph $V(MclG) = G\backslash\{1, 2, a, b, 3\}$ the *medium clique tree.* Call the medium clique tree without vertices c_i for $i \in I\backslash\{0\}$ on the underlying graph $V(SclG) = G\backslash(\{1, 2, a, b, 3\} \cup_{i \in I\backslash\{0\}} c_i)$ the *small clique tree.* From the small clique tree, form *tiny clique trees* τ_i for $i \in \overline{I}$ as follows. Remove all the arcs in the tooth containing c_0. Link c_0 to 4_i in both directions, and call the resulting clique tree τ_i. Have the vertices of the underlying graphs for these tiny clique trees be disjoint from each other except for each having c_0 and have the union of the vertex sets of these underlying graphs be the vertex set of the underlying graph for the small clique tree. Note that by assumption, except for the large clique tree, each of these clique trees either defines a facet on its respective underlying graph, or is the

exceptional comb on a 6 vertex underlying graph, which doesn't define a facet[5].

3.2 implementation of dimension reduction technique

This proof that any clique tree in $C_{=k+1} \cap T \cap S$ defines a facet consists of a sufficiently long valid list of equations and points. Recall from page #6 that this list is the result of carrying out the necessary number of stages of the dimension reduction procedure, and that for each stage, an equation is determined in step (ii) and a tour is determined in step (iii). It is shown that each tour in step (iii) is a *clique tree tour* (i.e. satisfies the clique tree inequality at equality). The other conditions that it must satisfy are easy to check. These conditions are that it satisfy the rest of the equations for the previous stages and that it violates the equation of its own stage. So the dimension reduction technique, when used to determine that clique trees are facets, proceeds as follows:

Start with P being the asymmetric traveling salesman polytope for an n vertex ATSP instance. Choose a clique tree inequality that you wish to show is facet-defining for step (ii) of stage 1.

At stage $j+1$ do the following:

(i) If P_j consists of a polytope of known dimension, such as the polytope of all clique tree tours of a clique tree that is known by induction to define a facet, then stop the procedure, with P_j being the *final polytope* for the procedure. Compute the lower bound on the dimension of the asymmetric traveling salesman polytope that results from this technique. If this lower bound coincides with the actual dimension of the asymmetric traveling salesman polytope, then the clique tree inequality in step (ii) of stage 1 has been shown to be facet-defining.

(ii) Otherwise choose an inequality $a_{j+1}x \leq b_{j+1}$ which is valid and supporting for P_j and define P_{j+1} by

$$P_{j+1} = \{x \in P_j \mid a_{j+1}x = b_{j+1}\}$$

(iii) Find a tour $x_{j+1} \in P_j$ such that $a_{j+1}x_{j+1} < b_{j+1}$. The tour which is used for this step, and hence satisfies the above conditions is referred to as the *desired tour.* In particular this is a clique tree tour for the clique tree introduced in step (ii) of stage 1. Increment j and proceed to the next stage.

To construct the tours in step (iii) , one starts with a clique tree tour on one of the smaller clique trees described above that violates some chosen equation (an equation of your choosing, not the equation of step (ii)). This is always possible when these clique trees are facets since then there is no equation which is valid for all clique tree tours (unless the equation is a linear combination of the clique tree equation and the degree constraints). For the case when one of the smaller clique trees is the comb on a 6 vertex underlying graph, one must explicitly show that there is a clique tree tour which violates one's chosen equation. This is done in section 3.9 on page #18. If the chosen equation is for instance $x_{6e} + x_{e6} = 1$ then one might want a clique tree tour such that $x_{6e} + x_{e6} \neq 1$, or equivalently, $x_{6e} + x_{e6} = 0$. Then one alters this tour in some way, such as replacing an arc with a path, obtaining an *altered tour.* This tour is then shown to be a clique tree tour, and is also shown to satisfy the other conditions that it must satisfy to be the desired tour that is used in step (iii).

3.3 determining whether the altered tour is a clique tree tour

Recall that an arc counts if it is in one of the cliques of the clique tree, so that using it in a tour increases the right hand side of the clique tree inequality by one for a simple clique tree. For this altered tour, the arcs which count that have been added from the clique tree tour one started with are shown and the number of such arcs is given. Also, the arcs which count that have been removed from the clique tree tour one started

with are shown and the number of these arcs is given. This is then compared with the difference in the right hand sides of the clique tree inequalities between the smaller clique tree and the large clique tree. If the net number of arcs which count that have been added from the clique tree tour one started with equals this difference, then the altered tour is a clique tree tour for the large clique tree. Note that this works because an arc which counts for one of the smaller clique trees also counts for the large clique tree.

3.4 needed facts about clique trees used in first part of proof

From page #3, note that when H_i for $i \in \{1..r\}$ are the handles and T_j for $j \in \{1..s\}$ are the teeth, the right hand side of the clique tree inequality is:

$$\sum_{i\in\{1..r\}} |H_i| + \sum_{j\in\{1..s\}} (|T_j| - t_j) - (s+1)/2$$

where t_j is the number of handles that tooth T_j intersects.

The big clique tree has one less handle and two fewer free teeth ($t_j = 1$) than the large clique tree. Hence the difference in the right hand sides between the large clique tree and the big clique tree is 3.

The medium clique tree also has one less node in one of its teeth. Hence the difference in the right hand sides between the large clique tree and the medium clique tree is 4.

The small clique tree has an additional $|I| - 1$ fewer nodes in this tooth. Hence, for the small clique tree this difference is $4 + |I| - 1$.

Recall from page #11 how the tiny clique tree τ_i for $i \in \overline{I}$ are formed. Denote the number of teeth in τ_i by s_i. Denote the number of teeth in the small clique tree by s_{sm}. Then, we have

$$\sum_{i\in\overline{I}} s_i = s_{sm} + |\overline{I}| - 1$$

From this, we get

$$\sum_{i \in \overline{I}} (s_i + 1)/2 = (s_{sm} - 1)/2 + |\overline{I}|$$

In converting from the small clique tree to all the tiny clique trees, the only handle or tooth which is disturbed in this process so as to affect the difference between the sum of the right hand sides for the tiny clique tree and the right hand side for the small clique tree is the tooth of each tiny clique tree τ_i for $i \in \overline{I}$ whose set of vertices is $\{4_i, c_0\}$. Denote the tooth in the small clique tree which contains c_0 by T_1. Now, T_1 intersects $|\overline{I}|$ handles, so it contributes only one to the right hand side for the small clique tree in the sense that $|T_1| - t_1 = 1$. On the other hand, each tooth $\{4_i, c_0\}$ in the tiny clique tree τ_i contributes one to the right hand side for that tiny clique tree. So, the sum of these contributions for all the tiny clique trees minus the contribution from the small clique tree is $|\overline{I}| - 1$. Now,

$$(|\overline{I}| - 1) - ((s_{sm} - 1)/2 + |\overline{I}|) = -(s_{sm} + 1)/2$$

Hence, the difference between the right hand sides of the large clique tree and the sum of the right hand sides of the tiny clique trees is also $4 + |I| - 1$.

3.5 strategy behind proof

One needs to understand the strategy behind this proof in order to more easily see that all the other conditions required for a valid proof are met by the altered tour created in step (iii) for each stage.

The basic strategy is as follows. One takes the starting polytope to be the ATSP polytope for the complete digraph underlying the large clique tree, and the next polytope P_1 as the convex hull of all tours on this digraph which are clique tree tours for the large clique tree.

If the vertices $a, 1, 2$, and b are removed from the large clique tree, what remains is the big clique tree. In particular, if a tour τ includes the path from a to 1 to 2 to b, then τ is a clique tree tour for the large clique tree if and only if τ is a clique tree tour for the big clique tree.

So one goes through as many intermediate nested polytopes as one can to arrive at the final polytope which is the convex hull of all tours τ which include the path from a to 1 to 2 to b and are clique tree tours for the big clique tree.

But this is a polytope of known dimension. That is, the vertices $a, 1, 2,$ and b can be replaced by a single node. Furthermore, the big cliquetree on the resulting underlying graph is by induction assumed to define a facet. But this means that the dimension of the final polytope is one less than the dimension of the asymmetric traveling salesman polytope for this underlying graph.

Furthermore, only equations which are implied by the system of equations:

$$\begin{aligned} x_{a1} &= 1 \\ x_{12} &= 1 \\ x_{2b} &= 1 \end{aligned}$$

can be used in step (ii) for the intermediate polytopes, since the final polytope is precisely P_1 with the three equations above added as constraints. Hence only arcs leaving a, entering b, leaving or entering 1 or 2, or the arc (b, a) can be constrained by the equations in step (ii) for each stage. These equations correspond to valid inequalities set at equality, and the two types of inequalities that are used in this proof are the trivial facets of the ATSP polytope, such as $x_{a1} \geq 0$ and the two city subtours, such as $x_{1a} + x_{a1} \leq 1$.

3.6 determining whether the altered tour satisfies the other conditions

In order to assist the reader in verifying that the altered tour created in step (iii) for each stage satisfies all the other conditions required by the dimension reduction procedure, a bipartite graph to be described below is shown after step (iii) for each stage from the middle phase of the proof onwards.

These other conditions are that the altered tour satisfies all the equations for the

previous stages besides just the clique tree equation, and that it violates the equation of its own stage.

The bipartite graph after step (iii) encodes what arcs the altered tour is allowed to use, and which of these arcs that it actually uses. In this proof an arc (i,j) is excluded from being used in step (iii) tours in future stages by an equation $x_{ij} = 0$ of step (ii).

The bipartite graph initially represents all the arcs which at some point in the proof might get excluded, but as of the beginning of the middle phase, haven't yet been excluded. Arcs that fit this description are those that leave a, those that enter b, those that leave or enter 1 or 2, or the arc (b,a). Each such arc (i,j) is represented in the bipartite graph by an edge (i,j) between node i on the left side of the bipartite graph and node j on the right side.

On the left side of the bipartite graph are certainly nodes a,1, and 2 since any arc leaving any of these three nodes can potentially get excluded at some stage. The other nodes on the left side of the bipartite graph are there so that arcs which enter the nodes b,1, and 2,(and hence can potentially get excluded at some stage), can be represented. The right side of the bipartition is similarly constructed. Node u that occurs on each side of the bipartition is the representation of any node which isn't otherwise represented on that side of the bipartition.

At each stage, the arcs which are still allowed to be used are represented by the edges which are still in the bipartite graph. If the equation for step (ii) at some stage was $x_{ij} = 0$, then arc (i,j) must be used in the tour of step (iii) for that stage, but that will be the last stage in which arc (i,j) can be used. This is indicated by a dotted edge in the bipartite graph. For all future bipartite graphs, this edge will then not be in the graph at all. The arcs used in the tour of step (iii) that leave a, enter b, or leave or enter 1 or 2, are shown by making the corresponding edges of the bipartite graph extra thick.

Whether constraints on the tours of step (iii) imposed by equations of the form

$x_{ij} + x_{ji} = 1$ are being satisfied has to be kept track of separately.

3.7 notational comments

When describing the tour of step (iii) for each stage, a piece of this tour will be explicitly shown by a list of nodes with each node linked to the next node on the list. When such links are arcs which count, a $\rightarrowtriangle$ is placed between the nodes in question, else a $\rightarrow$ is placed between the nodes in question. Also, the notation $c_0 \rightarrowtriangle \rightarrowtriangle c_i$ represents a path that starts at c_0 and goes through the rest of the vertices c_i for $i \in I \backslash \{0\}$, and $c_j \rightarrowtriangle \rightarrowtriangle c_i$ represents a path that starts at a vertex c_j just specified and goes through the rest of the vertices c_i for $i \in I \backslash \{j\}$. Note that all the arcs on these paths count.

3.8 numbering and counting

In the main body of the proof, rather than proceeding from stage to stage, the stages can be naturally grouped together. The proof explicitly shows each stage group, which consists of some definite number of stages, which are carried out in some arbitrary order. These stage groups are numbered, and the number of stages in each group is specified next to the stage group number.

Although there are more sophisticated ways of knowing whether enough stages have been carried out to show that our inequality defines a facet, one may simply add the number of stages for all the stage groups and check if this number is sufficiently large.

3.9 clique tree tours for 6 vertex comb

When one has a comb on an underlying graph of 6 vertices, since it is not a facet[5], there is a question as to whether one can find a clique tree tour which violates a given equation (see 3.2). But in this proof, it turns out that these equations are all of the form $x_{ij} + x_{ji} = 1$ or $x_{ij} = 0$ with one exception. In other words one is looking for clique tree tours such that $x_{ij} + x_{ji} \neq 1$, i.e. $x_{ij} + x_{ji} = 0$, or $x_{ij} \neq 0$, i.e. $x_{ij} = 1$. But finding such tours can easily be done for the comb on an underlying graph of 6

vertices since the equations that all clique tree tours for this comb must satisfy are not this simple.

Now we consider the exception, where a lemma is used to show the existence of a particular type of clique tree tour (see page #21). Let the handle of the comb on an underlying graph of 6 vertices be $\{4,5,6\}$ and the teeth be $\{c,4\},\{d,5\}$, and $\{e,6\}$. Then the clique tree tour

$$e \not\longrightarrow 6 \not\longrightarrow 5 \not\longrightarrow d \longrightarrow c \not\longrightarrow 4 \longrightarrow e$$

satisfies the conditions of this lemma.

3.10 main body of first part of proof

Recall from 3.2 that this proof consist of a list, which is the result of carrying out the necessary number of stages of the dimension reduction procedure. Furthermore, for each stage, an equation is determined in step (ii) and a tour is determined in step (iii). The stages are grouped into stage groups as described in 3.8.

3.10.1 beginning phase

The proof that the clique tree inequality corresponding to the large clique tree defines a facet goes as follows:

stage group 1: 1 stage

step (ii) : $\sum_{i\in\{1..r\}} x(E(H_i)) + \sum_{j\in\{1..s\}} x(E(T_j)) = \sum_{i\in\{1..r\}} |H_i| + \sum_{j\in\{1..s\}} (|T_j| - t_j) - (s+1)/2$

step(iii) : It is trivial to find a tour which doesn't satisfy the above equation.

stage group 2: $|I|$ stages

step (ii) : $x_{1c_j} = 0$ for $j \in I$

step(iii) : Consider the small clique tree with vertex c_j instead of vertex c_0. Suppose c_j is in a free tooth in this small clique tree. Since this small clique tree defines a facet, there exists a tour on these nodes that satisfies the equation $x_{c_j4} + x_{4c_j} = 0$. Replace

$\alpha \rightarrow c_j \rightarrow \beta$ in this tour by $\alpha \rightarrow a \rightarrowtriangle 1 \rightarrow c_j \rightarrowtriangle \rightarrowtriangle c_i \rightarrowtriangle 3 \rightarrowtriangle 2 \rightarrowtriangle b \rightarrow \beta$. Now $4 + |I| - 1$ new arcs that count have been created, and no arcs that count have been destroyed. By comparing this with the difference in the right hand sides of the two clique trees, one sees that the equation in step (ii) of the first stage has been satisfied by the new tour. Hence the new tour is a clique tree tour.

Suppose c_j isn't in a free tooth in this small clique tree. Consider all the tiny clique trees with vertex c_j instead of vertex c_0. For each of these tiny clique trees, find a tour for which neither of the arcs incident to c_j count, but which satisfies its corresponding clique tree inequality at equality. For all but one of these clique trees, remove c_j from the tour, leaving just a path. For an arbitrarily chosen tiny clique tree, replace $\alpha \rightarrow c_j \rightarrow \beta$ in the given tour with $\alpha \rightarrow a \rightarrowtriangle 1 \rightarrow c_j \rightarrowtriangle \rightarrowtriangle c_i \rightarrowtriangle 3 \rightarrowtriangle 2 \rightarrowtriangle b \rightarrow \beta$ and delete the arc coming into β to get a path again. Concatenate these paths and link the endpoints to form the desired tour. Note that the right hand side for the large clique tree inequality is the sum of the right hand sides for the tiny clique tree inequalities plus $4 + |I| - 1$. Furthermore none of the arcs which count for the tiny clique tree inequalities have been destroyed in the desired tour, but $4 + |I| - 1$ arcs which count for the large clique tree inequality have been added. So the desired tour is a clique tree tour.

stage group 3: $|I|$ stages

(ii) $x_{c_j 1} = 0$

(iii) Take the reverse tour from step (iii) of previous stage.

stage group 4: $|I|$ stages

(ii) $x_{2 c_j} = 0$

(iii) Construct the desired tour as in step (iii) of stage group 2, but replace $\alpha \rightarrow c_j \rightarrow \beta$ in the arbitrarily chosen tiny clique tree tour with $\alpha \rightarrow b \rightarrowtriangle 2 \rightarrow c_j \rightarrowtriangle \rightarrowtriangle c_i \rightarrowtriangle 3 \rightarrowtriangle 1 \rightarrowtriangle a \rightarrow \beta$.

stage group 5: $|I|$ stages

(ii) $x_{c_j 2} = 0$

(iii) Reverse the previous tour.

Now, stage groups 2 through 5 are repeated for all $j \in I$. Hence there are $4|I| + 1$ stages in all up to this point.

stage group 6: $\overline{I}$ stages

(ii) $x_{4_{\overline{j}}2} = 0$ for $\overline{j} \in \overline{I}$

(iii) Again consider the tiny clique trees. In choosing the starting tour which gets altered to form the desired tour, we need the following lemma.

LEMMA: Given a clique tree where c is in a tooth of size 2 and not in any handle, then there exists a tour which includes the path $\beta \dashrightarrow c \dashrightarrow 4 \dashrightarrow \alpha$ (with 4 being the other node in c's tooth) which satisfies the clique tree inequality at equality if it is facet-defining but where the first and last arcs of this path don't count.

PROOF: Suppose that the clique tree inequality is facet-defining but there is no such tour. Then $\sum_{h \text{ in 4's handle}} x_{4h} = 0$ implies ($x_{4c} = 1$ or $\sum_{4\alpha \text{ doesn't count}} x_{4\alpha} = 1$) which implies $x_{c4} = 0$ since it is assumed that no tour has the properties stated in the lemma.

We now need to show that for any tour which satisfies the clique tree inequality at equality, that $\sum_{h \text{ in 4's handle}} x_{4h} = 0$ implies $x_{c4} + x_{4c} = 1$. Suppose $\sum_{h \text{ in 4's handle}} x_{4h} = 0$ and $x_{c4} + x_{4c} = 0$ for such a tour. Call the vertex which comes after 4 in this tour as α. Repositioning vertex c between 4 and α then violates the clique tree inequality, which is a contradiction. Hence $\sum_{h \text{ in 4's handle}} x_{4h} = 0$ implies $x_{c4} + x_{4c} = 1$ for any tour which satisfies the clique tree inequality at equality.

Hence for any tour which satisfies the clique tree inequality at equality, then $\sum_{h \text{ in 4's handle}} x_{4h} = 0$ implies $x_{4c} = 1$. Also, $\sum_{h \text{ in 4's handle}} x_{4h} = 1$ implies $x_{4c} = 0$. Hence $\sum_{h \text{ in 4's handle}} x_{4h} + x_{4c} = 1$, which contradicts the assumption that the clique tree inequality was facet-defining. Hence there is a tour having the properties stated in this lemma.

So let a tour on $\tau_{\bar{j}}$ which includes the path $\beta \longrightarrow c_0 \rightarrowtriangle 4_{\bar{j}} \longrightarrow \alpha$ (with $4_{\bar{j}}$ being the other node in c_0's tooth) which satisfies the clique tree inequality at equality but where the first and last arcs of the path don't count be given. Replace arc $4_{\bar{j}}\alpha$ with $4_{\bar{j}} \longrightarrow 2 \rightarrowtriangle b \longrightarrow \alpha$ and replace arc βc_0 with $\beta \longrightarrow a \rightarrowtriangle 1 \rightarrowtriangle 3 \rightarrowtriangle c_i \rightarrowtriangle \rightarrowtriangle c_0$. Remove the link βa to obtain a path.

For $i \neq \bar{j}$ let a tour on τ_i be given which satisfies the clique tree inequality at equality but where $x_{c_0 4_i} + x_{4_i c_0} = 0$. Take out vertex c_0 from this tour, leaving a path.

Concatenate these paths together and link the ends to create the desired tour. None of the arcs which count in the tiny clique trees were destroyed to create the desired tour. Moreover, $4 + |I| - 1$ arcs which count have been added. Hence by the relationship of the right hand sides of these clique tree inequalities which was described above, the desired tour is a clique tree tour.

stage group 7: $\bar{I}$ stages

(ii) $x_{24_{\bar{j}}} = 0$

(iii) Reverse the previous tour.

stage group 8: $\bar{I}$ stages

(ii) $x_{4_{\bar{j}}1} = 0$

(iii) Construct the desired tour as in step (iii) of stage group 6, but for the arbitrarily chosen tiny clique tree tour which includes the path $\beta \longrightarrow c_0 \rightarrowtriangle 4_{\bar{j}} \longrightarrow \alpha$ do the following. Replace arc $4_{\bar{j}}\alpha$ with $4_{\bar{j}} \longrightarrow 1 \rightarrowtriangle a \longrightarrow \alpha$ and replace arc βc_0 with $\beta \longrightarrow b \rightarrowtriangle 2 \rightarrowtriangle 3 \rightarrowtriangle c_i \rightarrowtriangle \rightarrowtriangle c_0$.

stage group 9: $\bar{I}$ stages

(ii) $x_{14_{\bar{j}}} = 0$

(iii) Reverse the previous tour.

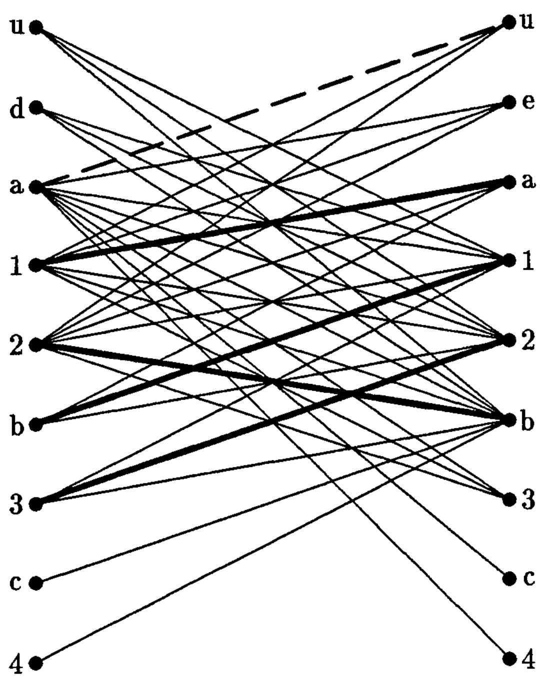
Repeat the last four stages for all $\bar{j} \in \bar{I}$. Up to this point $4|I| + 4|\bar{I}| + 1$ stages have been completed.

3.10.2 middle phase

stage group 10: 2 stages

(ii) $x_{as} = 0$ for $s = 5, d$.

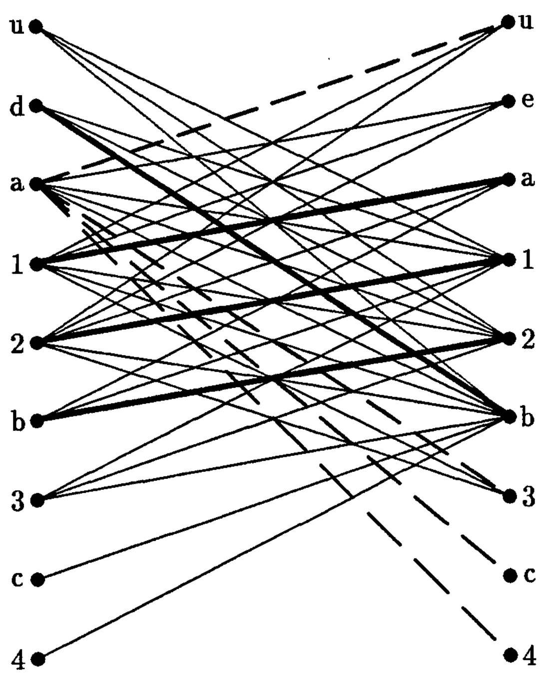
(iii) Consider the medium clique tree. Find a clique tree tour which contains the arc c_0s. Replace arc c_0s with $c_0 \nrightarrow 3 \nrightarrow 2 \nrightarrow b \dashrightarrow 1 \nrightarrow a \dashrightarrow s$ to form the desired tour. Since none of the arcs which count were destroyed and 4 arcs which count have been added, the desired tour is a clique tree tour.



stage group 11: $n - 7$ stages

(ii) $x_{at} = 0$ for $t \in \mathrm{V(BclG)} \setminus \{5, d, e\}$

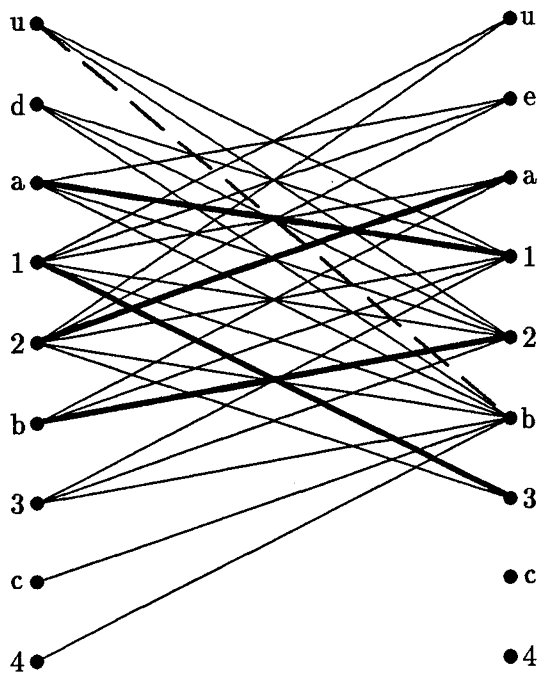
(iii) Consider a clique tree tour on the big clique tree which has the arc dt. Replace arc dt with $d \longrightarrow b \longrightarrow 2 \longrightarrow 1 \longrightarrow a \longrightarrow t$ to obtain the desired tour. Since 3 arcs which count have been added and none were destroyed, the desired tour is a clique tree tour.



stage group 12: 2 stages

(ii) $x_{sb} = 0$ for $s = 6, e$

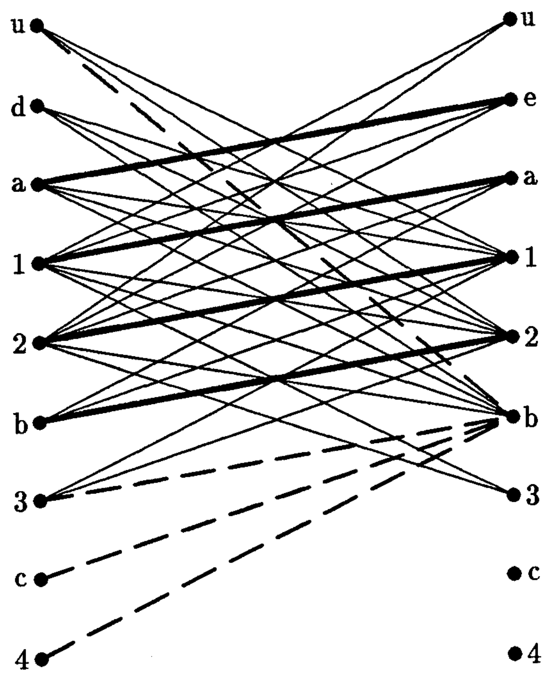
(iii) Replace arc sc_0 in a medium clique tree tour with $s \rightarrow b \nrightarrow 2 \rightarrow a \nrightarrow 1 \nrightarrow 3 \nrightarrow c_0$.



stage group 13: $n-7$ stages

(ii) $x_{tb}=0$ for $t \in \mathrm{V(BclG)} \backslash \{6, d, e\}$

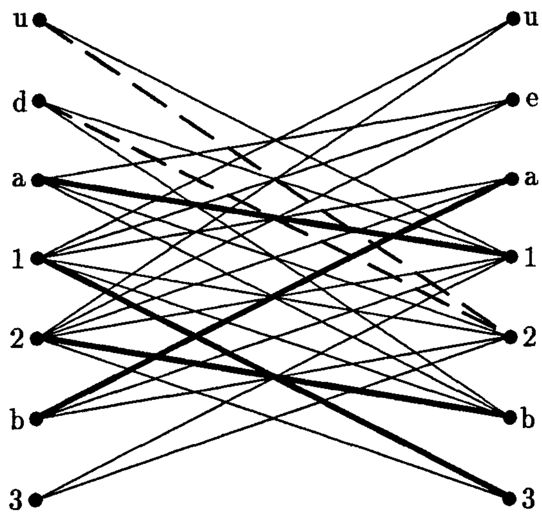
(iii) Replace arc te in a big clique tree tour with $t \longrightarrow b \mathrel{-\!\!\rhd} 2 \mathrel{-\!\!\rhd} 1 \mathrel{-\!\!\rhd} a \longrightarrow e$.



stage groups 14 and 15: $n - 5 - |I| - |\overline{I}|$ stages each

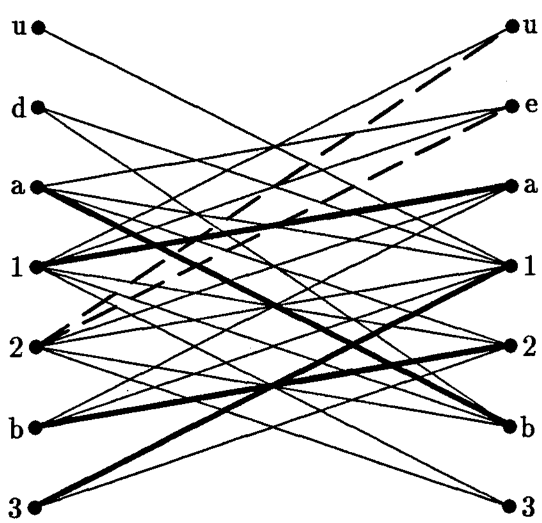
(ii) $x_{s2} = 0$ for $s \in \mathrm{V(SclG)} \backslash (\{c_0\} \cup \{4_{\overline{j}} \mid \overline{j} \in \overline{I}\})$

(iii) Replace arc $s3$ in a big clique tree tour with $s \twoheadrightarrow 2 \rightarrowtriangle b \twoheadrightarrow a \rightarrowtriangle 1 \rightarrowtriangle 3$.



(ii) $x_{2s} = 0$ for $s \in \mathrm{V(SclG)} \backslash (\{c_0\} \cup \{4_{\overline{j}} \mid \overline{j} \in \overline{I}\})$

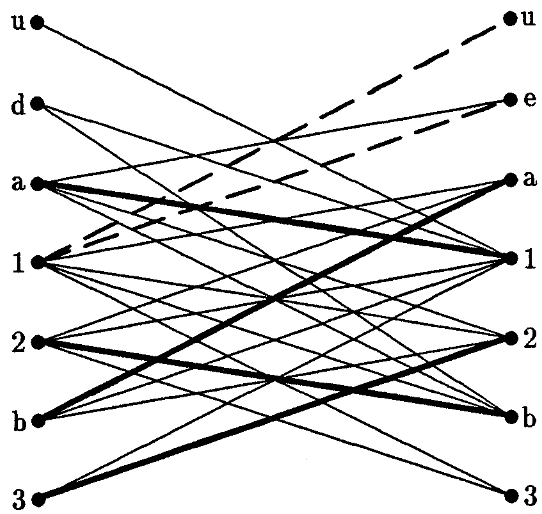
(iii)reverse the previous tour.



stage groups 16 and 17: $n - 5 - |I| - |\overline{I}|$ stages each

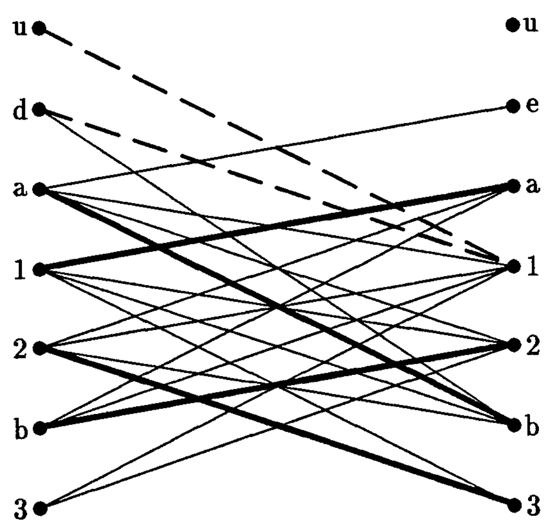
(ii) $x_{1s} = 0$ for $s \in$ V(SclG)$\backslash(\{c_0\} \cup \{4_{\overline{j}} \,|\, \overline{j} \in \overline{I}\})$

(iii)Replace arc $3s$ in a big clique tree tour with $3 \twoheadrightarrow 2 \twoheadrightarrow b \rightarrow a \twoheadrightarrow 1 \rightarrow s$.



(ii) $x_{s1} = 0$ for $s \in$ V(SclG)$\backslash(\{c_0\} \cup \{4_{\overline{j}} \,|\, \overline{j} \in \overline{I}\})$

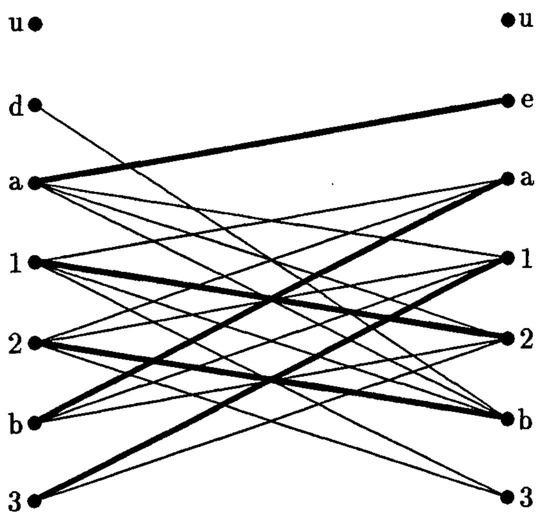
(iii) Reverse the previous tour.



stage 18

(ii) $x_{1a} + x_{a1} = 1$

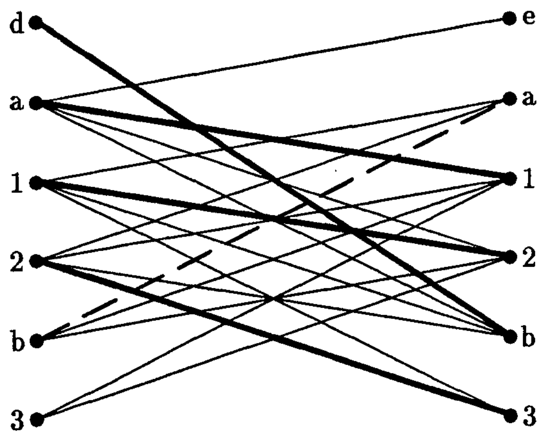
(iii) Replace arc $3e$ in a big clique tree tour with $3 \rightarrow 1 \rightarrow 2 \rightarrow b \dashrightarrow a \dashrightarrow e$



stage 19

(ii) $x_{ba} = 0$

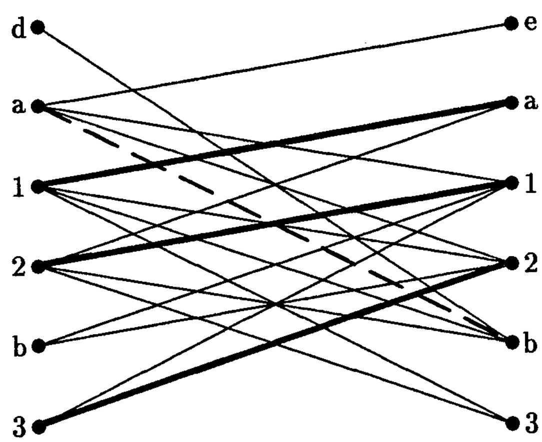
(iii) Replace arc $d3$ in a big clique tree tour with $d \twoheadrightarrow b \twoheadrightarrow a \twoheadrightarrow 1 \twoheadrightarrow 2 \twoheadrightarrow 3$.



stage 20

(ii) $x_{ab} = 0$

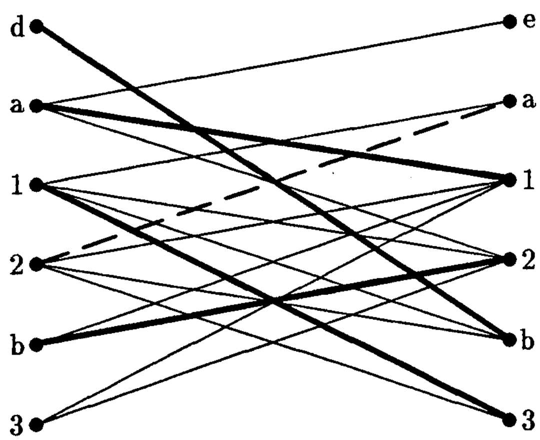
(iii) Replace arc $3d$ in a big clique tree tour with $3 \twoheadrightarrow 2 \twoheadrightarrow 1 \twoheadrightarrow a \twoheadrightarrow b \twoheadrightarrow d$



stage 21

(ii) $x_{2a} = 0$

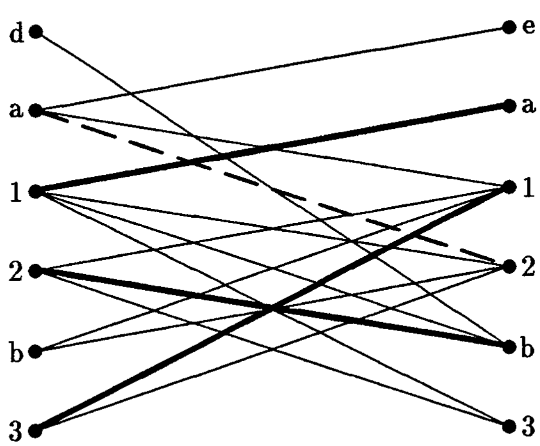
(iii) Replace arc $d3$ in a big clique tree tour with $d \rightarrow b \rightarrow 2 \rightarrow a \rightarrow 1 \rightarrow 3$.



stage 22

(ii) $x_{a2} = 0$

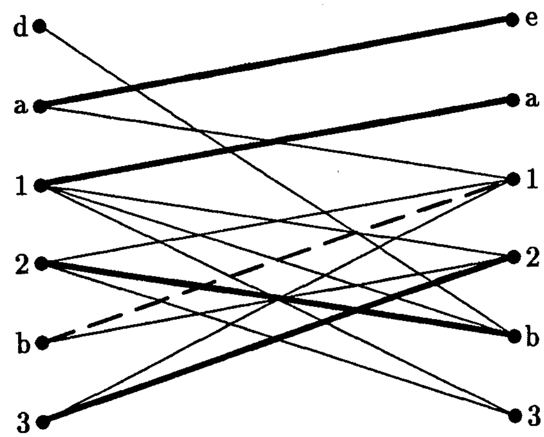
(iii) Reverse the previous tour.



stage 23

(ii) $x_{b1} = 0$

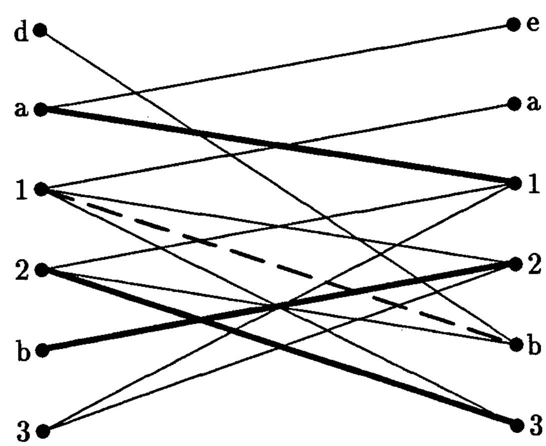
(iii) Replace arc $3e$ in a big clique tree tour with $3 \twoheadrightarrow 2 \twoheadrightarrow b \rightarrow 1 \twoheadrightarrow a \rightarrow e$.



stage 24

(ii) $x_{1b} = 0$

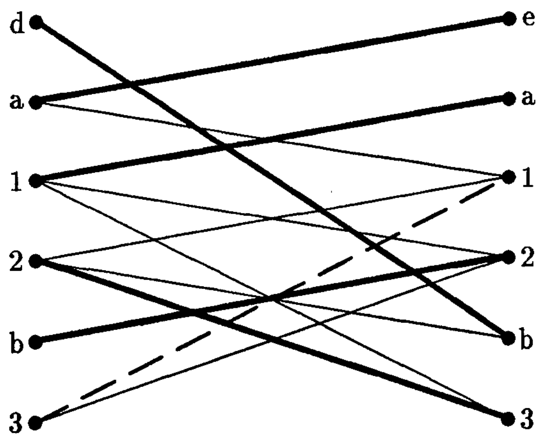
(iii) Reverse the previous tour.



stage 25

(ii) $x_{31} = 0$

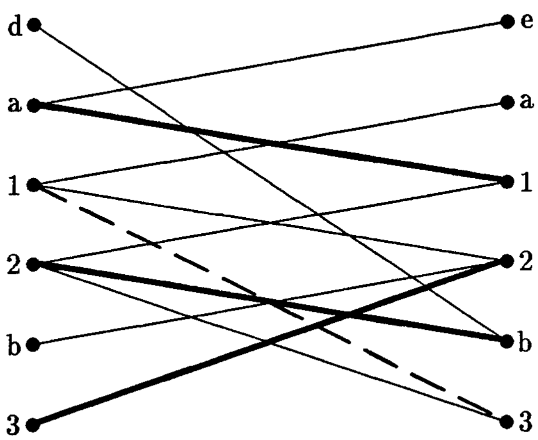
(iii) Replace arc de in a medium clique tree tour with $d \rightarrow b \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow a \rightarrow e$.



stage 26

(ii) $x_{13} = 0$

(iii) Reverse the previous tour.

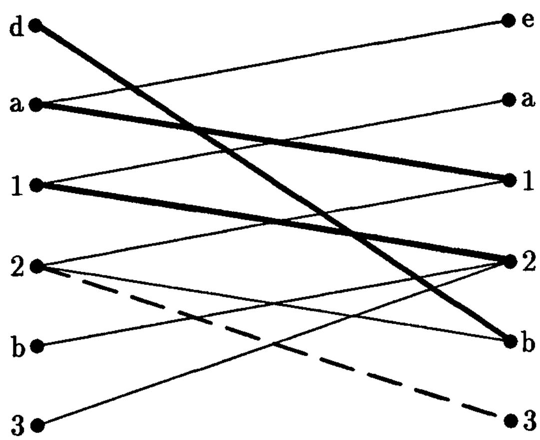


3.10.3 ending phase

stage 27

(ii) $x_{23} = 0$

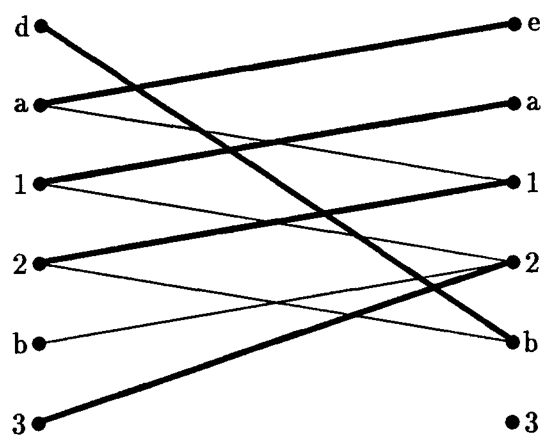
(iii) Choose a clique tree tour on the tiny clique tree that has vertex e such that $x_{6e} + x_{e6} = 0$. A maximal path of arcs which count that contains c_0 either starts or ends at c_0 If it ends at c_0 then reverse it. Then place this path right after e in the tour. If the maximal path of arcs which count that contains d starts at d then reverse it. Place node b after d and replace arc ec_0 with $e \rightarrow a \rhd 1 \rhd 2 \rhd 3 \rhd c_i \rhd \rhd c_0$. Remove the link right after b to form a path. For the other tiny clique trees, pick a clique tree tour satisfying $x_{c_0 4_i} + x_{4_i c_0} = 0$. Remove c_0, forming a path. Concatenate all these paths and link the ends together. This yields the desired tour.



stage 28

(ii) $x_{b2} + x_{2b} = 1$

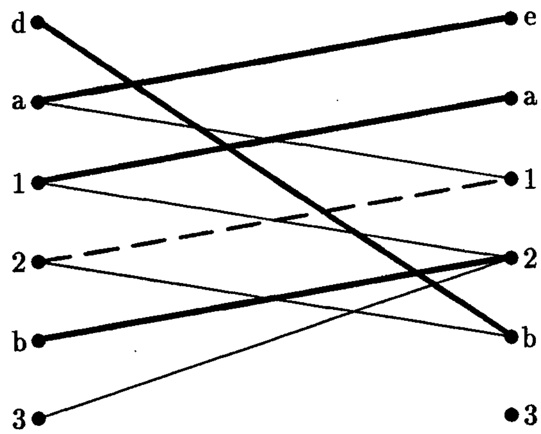
(iii) Choose a clique tree tour on the tiny clique tree $\tau_{\bar{j}}$ that has vertex e such that $x_{c_0 4_{\bar{j}}} + x_{4_{\bar{j}} c_0} = 0$. A maximal path of arcs which count that contains e either starts or ends at e. If it ends at e then reverse it. Then place this path right after c_0 in the tour. If the maximal path of arcs which count that contains d starts at d then reverse it. Place node b after d and replace arc $c_0 e$ with $c_0 \twoheadrightarrow \twoheadrightarrow c_i \twoheadrightarrow 3 \twoheadrightarrow 2 \twoheadrightarrow 1 \twoheadrightarrow a \rightarrow e$. Remove the link right after b to form a path. For the other tiny clique trees, pick a clique tree tour satisfying $x_{c_0 4_i} + x_{4_i c_0} = 0$. Remove c_0, forming a path. Concatenate all these paths and link the ends together. This yields the desired tour.



stage 29

(ii) $x_{21} = 0$

(iii) Replace arc de in a big clique tree tour with $d \rightarrow b \rightarrow 2 \rightarrow 1 \rightarrow a \rightarrow e$



3.10.4 evaluation of results

Now the polytope that satisfies the equations in step (ii) for all the stages is the one where $x_{a1} = x_{12} = x_{2b} = 1$ and the clique tree inequality for the big clique tree is set at equality. Without changing the dimension of this polytope, $a, 1, 2,$ and b can be contracted into a single node, yielding $n - 3$ nodes, where n is the original number of vertices. The dimension of the asymmetric traveling salesman polytope is $n^2 - 3n + 1$. Therefore, the dimension of this polytope is $(n-3)^2 - 3(n-3) + 1 - 1$ since by the inductive assumption, the clique tree inequality for the big clique tree defines a facet, especially since the node $a, 1, 2,$ and b was contracted to isn't in the big clique tree. Since there were $6n - 17$ stages in this proof in all, this gives a lower bound on the dimension of the asymmetric traveling salesman polytope of $n^2 - 3n + 1$. Since this is its actual dimension, the clique tree inequality for the large clique tree defines a facet.

Hence since the large clique tree and the underlying graph G were arbitrary, then for any underlying graph G, all clique trees in $C_{=k+1} \cap T \cap S$ define facets.

Part II
second part of proof

4 type of clique tree under investigation

Assume all clique trees in $C_k \cup (C_{=k+1} \cap T \cap H_l)$ define facets on any underlying graph. Let an underlying graph G be given. Let a clique tree in $C_{=k+1} \cap T \cap H_{l+1}$ on the graph G be given. Call this the large clique tree. Label the vertices of the special handle $1, 2, 3$ and $1_i, 2_i$ for $i \in L$. Have 3 be the vertex that intersects the non-free tooth. Let the free teeth intersecting this handle be $\{a, 1\}, \{b, 2\}$, and $\{a_i, 1_i\}, \{b_i, 2_i\}$ for $i \in L$. Note that $l = |L|$. Denote the nodes in the tooth containing 3 which aren't in any handle by c_i for $i \in I$. Denote the other nodes of this tooth by 4_i for $i \in \overline{I}$. Choose a special vertex c_0 from the family of vertices c_i for $i \in I$. Choose a handle other than the special handle that has two free teeth in this clique tree and denote the vertices of these teeth as $\{5, d\}$ and $\{6, e\}$, with 5 and 6 also being vertices in the handle.

4.1 clique trees used in second part

Call the given clique tree the large clique tree. Call the clique tree without vertices $1, 2, a$, and b on the underlying graph $V(BclG) = G \backslash \{1, 2, a, b\}$ the big clique tree. Call the clique tree without vertices $1, 2, a, b, 3, 1_i, 2_i, a_i$, and b_i for $i \in L$ on the underlying graph $V(MeclG) = G \backslash (\{1, 2, a, b, 3\} \cup \{1_i, 2_i, a_i, b_i \,|\, i \in L\})$ the medium clique tree. Call the medium clique tree without vertices c_i for $i \in I \backslash \{0\}$ on the underlying graph $V(SmclG) = V(MeclG) \backslash \{c_i \,|\, i \in I \backslash \{0\}$ the small clique tree. From the small clique tree, form tiny clique trees τ_i for $i \in \overline{I}$ as follows. Remove all the arcs in the tooth containing c_0. Link c_0 to 4_i in both directions, and call the resulting clique tree τ_i. Have the vertices of the underlying graphs for these tiny clique trees be disjoint from each other except for each having c_0 and have the union of the vertex sets of these underlying graphs be the vertex set of the underlying graph for the small clique tree. Note that by

assumption, these clique trees all define facets on their respective underlying graphs. Finally, define a zig-zag to be the path $a_1 \dashrightarrow 1_1 \dashrightarrow 2_1 \dashrightarrow b_1 \longrightarrow a_2 \dashrightarrow 1_2 \dashrightarrow \ldots \longrightarrow a_l \dashrightarrow 1_l \dashrightarrow 2_l \dashrightarrow b_l$.

4.2 implementation of dimension reduction technique

As before, the proof that any clique tree in $C_{=k+1} \cap T \cap H_{l+1}$ defines a facet consists of a sufficiently long valid list of equations and points. It is just a slight modification of the proof of the first part.

4.3 needed facts about clique trees used in the second part of proof

The difference in the right hand sides between the large clique tree and the big clique tree is 3. For the differences for the other clique trees, the number of arcs that count in a zig-zag, which is $3l$, comes into play. The difference in the right hand sides between the large clique tree and the medium clique tree is $4 + 3l$. The difference in the right hand sides between the large clique tree and the small clique tree is $4 + |I| - 1 + 3l$. Finally, the difference between the right hand sides of the large clique tree and the sum of the right hand sides of the tiny clique trees is also $4 + |I| - 1 + 3l$.

4.4 main body of second part of proof

4.4.1 beginning phase

So, here is the proof:

stage group 1: 1 stage

step (ii) : $\sum_{i \in \{1..r\}} x(E(H_i)) + \sum_{j \in \{1..s\}} x(E(T_j)) = \sum_{i \in \{1..r\}} |H_i| + \sum_{j \in \{1..s\}} (|T_j| - t_j) - (s+1)/2$

step(iii) : It is trivial to find a tour which doesn't satisfy the above equation.

stage group 2: $|I|$ stages

step (ii) : $x_{1c_j} = 0$ for $j \in I$

step(iii) : First, suppose the tooth which contains c_j intersects only one handle. Let a clique tree tour on the small clique tree that satisfies the equation $x_{c_j4} + x_{4c_j} = 0$ be the starting tour. Alter this tour by replacing $\alpha c_j \beta$ by $\alpha \to$ zig-zag $\to a \rightarrowtriangle 1 \rightarrowtriangle c_j \rightarrowtriangle\rightarrowtriangle c_i \rightarrowtriangle 3 \rightarrowtriangle 2 \rightarrowtriangle b \to \beta$. None of the arcs which counted in the starting tour were destroyed. Furthermore $4 + |I| - 1 + 3|T|$ new arcs that count have been created. Hence the altered tour is a clique tree tour on the large clique tree.

Suppose the tooth containing c_j intersects more than two handles. For each of these tiny clique trees τ_i, find a clique tree tour that doesn't use either $4_i c_j$ or $c_j 4_i$ for the starting tours. For all but one of these clique trees, remove c_j from the tour, leaving just a path. For an arbitrarily chosen clique tree, replace $\alpha \to c_j \to \beta$ in the given tour with $\alpha \to a \rightarrowtriangle 1 \rightarrowtriangle c_j \rightarrowtriangle\rightarrowtriangle c_i \rightarrowtriangle 3 \rightarrowtriangle 2 \rightarrowtriangle b$ zig-zag β and delete the arc coming into β to get a path again. Concatenate these paths and link the endpoints to form the desired tour. Note that the right hand side for the large clique tree inequality is the sum of the right hand sides for the tiny clique tree inequalities plus $4 + |I| - 1 + 3l$. Furthermore none of the arcs which count for any of the tiny clique trees have been destroyed in the desired tour, but $4 + |I| - 1 + 3l$ arcs which count for the large clique tree have been added. So the desired tour is a clique tree tour on the large clique tree.

stage group 3: $|I|$ stages

(ii) $x_{c_j1} = 0$

(iii) Take the reverse tour from step (iii) of previous stage.

stage group 4: $|I|$ stages

(ii) $x_{2c_j} = 0$

(iii) Using symmetry arguments, a tour like the tours for step (iii) of the two previous stages can be found.

stage group 5: $|I|$ stages

(ii) $x_{c_j2} = 0$

(iii) Reverse the previous tour.

Now, stages 2 through 5 are repeated for all $j \in I$. Hence there are $4|I|+1$ stages in all up to this point.

stage group 6: $|\overline{I}|$ stages

(ii) $x_{4_{\overline{j}}2} = 0$ for $\overline{j} \in \overline{I}$

(iii) So let a clique tree tour on $\tau_{\overline{j}}$ which includes the path $\beta \rightarrow c_0 \twoheadrightarrow 4_{\overline{j}} \rightarrow \alpha$ where the first and last arcs of this path don't count be given. Note that this is possible by the lemma in the first part of this proof. Alter this tour by replacing arc $4_{\overline{j}}\alpha$ with $4_{\overline{j}} \rightarrow 2 \twoheadrightarrow b$ zig-zag α and replacing arc βc_0 with $\beta \rightarrow a \twoheadrightarrow 1 \twoheadrightarrow 3 \twoheadrightarrow c_i \twoheadrightarrow \twoheadrightarrow c_0$. Remove the link βa to obtain a path.

For $i \neq \overline{j}$ let a clique tree tour on τ_i be given such that $x_{c_0 4_i} + x_{4_i c_0} = 0$. Take out vertex c_0 from this tour, leaving a path.

Concatenate these paths together and link the ends to create the desired tour. None of the arc which counted in the tiny clique tree tours were destroyed to create the desired tour. Moreover, $4+|I|-1+3l$ arcs which count have been added. Hence the desired tour is a clique tree tour on the large clique tree.

stage group 7: $|\overline{I}|$ stages

(ii)$x_{24_{\overline{j}}} = 0$

(iii) Reverse the previous tour.

stage group 8: $|\overline{I}|$ stages

(ii)$x_{4_{\overline{j}}1} = 0$

(iii) Using symmetry arguments, a tour like the tours for step (iii) of the two previous stages can be found.

stage group 9: $|\overline{I}|$ stages

(ii)$x_{14_{\overline{j}}} = 0$

(iii) Reverse the previous tour.

Repeat the last four stages for all $\overline{j} \in \overline{I}$ Up to this point $4|I|+4|\overline{I}|+1$ stages have been completed.

4.4.2 middle phase

stage group 10: 2 stages

(ii) $x_{as} = 0$ for $s = 5, d$

(iii) Have the starting tour be a clique tree tour on the medium clique tree which contains the arc $c_0 s$. To alter this tour, replace arc $c_0 s$ with $c_0 \not\dashrightarrow 3 \not\dashrightarrow 2 \not\dashrightarrow b$ zig-zag $1 \nrightarrow a \dashrightarrow s$ to form the desired tour. Since none of the arcs which count were destroyed and $4 + 3l$ arcs which count have been added, the desired tour is a clique tree tour on the large clique tree.

stage group 11: $n - 7$ stages

(ii) $x_{at} = 0$ for $t \in V(BclG)\backslash\{5, d, e\}$

(iii) Take as a starting tour a clique tree tour on the big clique tree which has the arc dt. Replace arc dt with $d \dashrightarrow b \nrightarrow 2 \nrightarrow 1 \nrightarrow a \dashrightarrow t$ to obtain the desired tour. Since 3 arcs which count have been added and none destroyed, the desired tour is a clique tree tour for the large clique tree.

stage group 12: 2 stages

(ii) $x_{sb} = 0$ for $s = 6, e$

(iii) Replace arc sc_0 in a medium clique tree tour with $s \dashrightarrow b \nrightarrow 2$ zig-zag $a \nrightarrow 1 \nrightarrow 3 \nrightarrow c_0$.

stage group 13: $n - 7$ stages

(ii) $x_{tb} = 0$ for $t \in V(BclG)\backslash\{6, d, e\}$

(iii) Replace arc te in a big clique tree tour with $t \dashrightarrow b \nrightarrow 2 \nrightarrow 1 \nrightarrow a \dashrightarrow e$.

stage group 14: $n - 5 - |I| - |\overline{I}|$ stages

(ii) $x_{s2} = 0$ for $s \in V(SmclG)\backslash(\{c_0\} \cup \{4_{\overline{j}} \,|\, \overline{j} \in \overline{I}\})$

(iii) Replace arc $s3$ in a big clique tree tour with $s \dashrightarrow 2 \nrightarrow b \dashrightarrow a \nrightarrow 1 \nrightarrow 3$. However, one must be careful here. If $s \in \{1_i \,|\, i \in L\} \cup \{2_i \,|\, i \in L\}$ then one should denote this as replacing $s \nrightarrow 3$ with $s \nrightarrow 2 \nrightarrow b \dashrightarrow a \nrightarrow 1 \nrightarrow 3$ since $s3$ and $s2$ both count.

stage group 15: $n-5-|I|-|\bar{I}|$ stages

(ii) $x_{2s}=0$ for $s \in V(SmclG)\backslash(\{c_0\} \cup \{4_{\bar{j}} \mid \bar{j} \in \bar{I}\})$

(iii) Reverse the previous tour.

stage group 16: $n-5-|I|-|\bar{I}|$ stages

(ii) $x_{1s}=0$ for $s \in V(SmclG)\backslash(\{c_0\} \cup \{4_{\bar{j}} \mid \bar{j} \in \bar{I}\})$

(iii) Replace arc $3s$ in a big clique tree tour with $3 -\!\!\triangleright 2 -\!\!\triangleright b \rightarrow a -\!\!\triangleright 1 \rightarrow s$.

stage group 17: $n-5-|I|-|\bar{I}|$ stages

(ii) $x_{s1}=0$ for $s \in V(SmclG)\backslash(\{c_0\} \cup \{4_{\bar{j}} \mid \bar{j} \in \bar{I}\})$

(iii) Reverse the previous tour.

stage 18

(ii) $x_{1a}+x_{a1}=0$

(iii) Replace arc $3e$ in a big clique tree tour with $3 -\!\!\triangleright 1 -\!\!\triangleright 2 -\!\!\triangleright b \rightarrow a \rightarrow e$

stage 19

(ii) $x_{ba}=0$

(iii) Replace arc $d3$ in a big clique tree tour with $d \rightarrow b \rightarrow a -\!\!\triangleright 1 -\!\!\triangleright 2 -\!\!\triangleright 3$.

stage 20

(ii) $x_{ab}=0$

(iii) Replace arc $3d$ in a big clique tree tour with $3 -\!\!\triangleright 2 -\!\!\triangleright 1 -\!\!\triangleright a \rightarrow b \rightarrow d$.

stage 21

(ii) $x_{2a}=0$

(iii) Replace arc $d3$ in a big clique tree tour with $d \rightarrow b -\!\!\triangleright 2 \rightarrow a -\!\!\triangleright 1 -\!\!\triangleright 3$.

stage 22

(ii)$x_{a2}=0$

(iii) Reverse the previous tour.

stage 23

(ii) $x_{b1}=0$

(iii) Replace arc $3e$ in a big clique tree tour with $3 -\!\!\triangleright 2 -\!\!\triangleright b \rightarrow 1 -\!\!\triangleright a \rightarrow e$.

stage 24

(ii)$x_{1b} = 0$

(iii) Reverse the previous tour.

stage 25

(ii) $x_{31} = 0$

(iii) Replace arc de in a medium clique tree tour with $d \rightarrow b \rightarrowtriangle 2 \rightarrowtriangle 3 \rightarrowtriangle 1 \rightarrowtriangle a \rightarrow e$ and insert zig-zag into another arc which doesn't count.

stage 26

(ii)$x_{13} = 0$

(iii) Reverse the previous tour.

4.4.3 ending phase

stage 27

(ii) $x_{23} = 0$

(iii) Choose a clique tree tour on the tiny clique tree that has vertex e such that $x_{6e} + x_{e6} = 0$. A maximal path of arcs which count that contains c_0 either starts or ends at c_0 If it ends at c_0 then reverse it. Then place this path right after e in the tour. If the maximal path of arcs which count that contains d starts at d then reverse it. Place node b and then zig-zag after d and replace arc ec_0 with $e \rightarrow a \rightarrowtriangle 1 \rightarrowtriangle 2 \rightarrowtriangle 3 \rightarrowtriangle c_i \rightarrowtriangle \rightarrowtriangle c_0$. Remove the link right after b to form a path. For the other tiny clique trees, pick a clique tree tour satisfying $x_{c_0 4_i} + x_{4_i c_0} = 0$. Remove c_0, forming a path. Concatenate all these paths and link the ends together. This yields the desired tour.

stage 28

(ii) $x_{b2} + x_{2b} = 1$

(iii) Choose a clique tree tour on the tiny clique tree $\tau_{\bar{j}}$ that has vertex e such that $x_{c_0 4_{\bar{j}}} + x_{4_{\bar{j}} c_0} = 0$. A maximal path of arcs which count that contains e either starts or ends at e. If it ends at e then reverse it. Then place this path right after

c_0 in the tour. If the maximal path of arcs which count that contains d starts at d then reverse it. Place node b and then zig-zag after d and replace arc c_0e with $c_0 \twoheadrightarrow \twoheadrightarrow c_i \twoheadrightarrow 3 \twoheadrightarrow 2 \twoheadrightarrow 1 \twoheadrightarrow a \rightarrow e$. Remove the link right after b to form a path. For the other tiny clique trees, pick a clique tree tour satisfying $x_{c_0 4_i} + x_{4_i c_0} = 0$. Remove c_0, forming a path. Concatenate all these paths and link the ends together. This yields the desired tour.

stage 29

(ii) $x_{21} = 0$

(iii) Replace arc de in a big clique tree tour with $d \rightarrow b \twoheadrightarrow 2 \twoheadrightarrow 1 \twoheadrightarrow a \rightarrow e$.

4.4.4 evaluation of results

Now the polytope that satisfies the equations in step (ii) for all the stages is the one where $x_{a1} = x_{12} = x_{2b} = 1$ and the clique tree inequality for the big clique tree is set at equality. Without changing the dimension of this polytope, $a, 1, 2,$ and b can be contracted into a single node, yielding $n - 3$ nodes, where n is the original number of vertices. The dimension of the asymmetric traveling salesman polytope is $n^2 - 3n + 1$. Therefore, the dimension of this polytope is $(n-3)^2 - 3(n-3) + 1 - 1$ since by the inductive assumption, the clique tree inequality for the big clique tree defines a facet. Since there were $6n - 17$ stages in this proof in all, this gives a lower bound on the dimension of the asymmetric traveling salesman polytope of $n^2 - 3n + 1$. Since this is its actual dimension, the clique tree inequality for the large clique tree defines a facet.

Hence since the large clique tree and the underlying graph G were arbitrary, then for any underlying graph G, all clique trees in $C_{=k+1} \cap T \cap H_{l+1}$ define facets. As was observed in the outline of the proof (section 2.4) by induction on l, one can conclude that for any underlying graph G, all clique trees in $C_{=k+1} \cap T \cap H$ define facets.

Part III
third part of proof

5 type of clique tree under investigation

Assume all clique trees in $C_k \cup (C_{=k+1} \cap T \cap H)$ define facets on any underlying graph. Let an underlying graph G be given. Let a clique tree in $C_{=k+1} \cap T \cap M$ on the graph G be given. Call this the large clique tree. Label the vertex in the special handle which isn't in any tooth as m. Label the other vertices of the special handle 1,2,3, and $1_i, 2_i$ for $i \in L$. Have 3 be the vertex that intersects the non-free tooth. The vertices in the teeth intersecting the special handle are the same as those described in this section for the second part of the proof. Similarly, as before, choose a handle other than the special handle that has two free teeth in this clique tree and denote the vertices of these teeth as $\{5, d\}$ and $\{6, e\}$, with 5 and 6 also being vertices in the handle.

5.1 clique trees used in third part

Call the given clique tree the large clique tree. Call the clique tree without node m on the underlying graph $G\backslash\{m\}$ the clique tree without node m. The other clique trees are defined as in the proof of part 2 but without vertex m in the clique tree or the underlying graph.

5.2 needed facts about clique trees used in third part of proof

The difference in the right hand sides between the large clique tree and the clique tree without node m is one. The differences in the right hand sides between the large clique tree and the other clique trees is in each case one greater than it was in the proof of part 2 due to the presence of vertex m in the large clique tree.

5.3 main body of third part of proof

5.3.1 beginning phase

So, here is the proof:

stage 1:

step (ii) : $\sum_{i\in\{1..r\}} x(E(H_i)) + \sum_{j\in\{1..s\}} x(E(T_j)) = \sum_{i\in\{1..r\}} |H_i| + \sum_{j\in\{1..s\}}(|T_j| - t_j) - (s+1)/2$

step(iii) : It is trivial to find a tour which doesn't satisfy the above equation.

stage group m1: $n-5$ stages

(ii) :$x_{mt} = 0$ for $t \in G\backslash\{1,2,3,a\}$

(iii) :Choose a clique tree tour on the clique tree without node m that includes arc $1t$. Replace this with $1 \rightarrow m \rightarrow t$ to get the desired tour. Note that arcs $1t$ and mt either both count or both don't count.

stage group m2: $n-5$ stages

(ii) :$x_{tm} = 0$ for $t \in G\backslash\{1,2,3,a\}$

(iii) :Use the reverse tours.

stage m3

(ii) :$x_{ma} = 0$

(iii) :Choose a clique tree tour on the clique tree without node m that includes arc $2a$. Replace this with $2 \rightarrow m \rightarrow a$ to get the desired tour.

stage m4

(ii)$x_{am} = 0$

(iii) Reverse the previous tour.

stage n1

(ii) :$x_{12} = 0$

(iii) :Choose a clique tree tour on the big clique tree that includes arc $e3$ which doesn't count. Replace this with $e \rightarrow b \rightarrow a \not\rightarrow 1 \not\rightarrow 2 \not\rightarrow m \not\rightarrow 3$

stage n2

(ii) :$x_{21} = 0$

(iii) :Choose a clique tree tour on the big clique tree that includes arc $e3$. Replace this with $e \dashrightarrow a \dashrightarrow b \twoheadrightarrow 2 \twoheadrightarrow 1 \twoheadrightarrow m \twoheadrightarrow 3$

stage group 2: $|I|$ stages

step (ii) : $x_{1c_j} = 0$ for $j \in I$

step(iii) :

For each tiny clique tree τ_i, find a clique tree tour that satisfies $x_{4_ic_j} + x_{c_j4_i} = 0$. For all but one of these clique trees, remove c_0 from the tour, leaving just a path. For an arbitrarily chosen clique tree, replace $\alpha \dashrightarrow c_j \dashrightarrow \beta$ in the given tour with $\alpha \dashrightarrow a \twoheadrightarrow 1 \dashrightarrow c_j \twoheadrightarrow\twoheadrightarrow c_i \twoheadrightarrow 3 \twoheadrightarrow m \twoheadrightarrow 2 \twoheadrightarrow b$ zig-zag β and delete the arc coming into β to get a path again. Concatenate these paths and link the endpoints to form the desired tour. So, none of the arcs which count for the tiny clique tree inequalities have been destroyed in the desired tour, but $5 + |I| - 1 + 3|L|$ arcs which count for the large clique tree inequality have been added. So the desired tour is a clique tree tour.

stage group 3: $|I|$ stages

(ii) $x_{c_j1} = 0$

(iii) Take the reverse tour from step (iii) of previous stage.

stage group 4: $|I|$ stages

(ii) $x_{2c_j} = 0$

(iii) Using symmetry arguments, a tour like the tours for step (iii) of the two previous stages can be found.

stage group 5: $|I|$ stages

(ii) $x_{c_j2} = 0$

(iii) Reverse the previous tour.

Now, the last four stages are repeated for all $j \in I$.

stage group 6: $|\overline{I}|$ stages

(ii) $x_{4_{\overline{j}}2} = 0$ for $\overline{j} \in \overline{I}$

(iii) So let a clique tree tour on $\tau_{\overline{j}}$ which includes the path $\beta \dashrightarrow c_0 \dashrightarrow 4_{\overline{j}} \dashrightarrow \alpha$ where the first and last arcs of the path don't count be given. This is possible by the lemma from the first part of the proof. Replace arc $4_{\overline{j}}\alpha$ with $4_{\overline{j}} \dashrightarrow 2 \dashrightarrow b$ zig-zag α and replace arc βc_0 with $\beta \dashrightarrow a \dashrightarrow 1 \dashrightarrow m \dashrightarrow 3 \dashrightarrow c_i \dashrightarrow \dashrightarrow c_0$. Remove the link βa to obtain a path.

For $i \neq \overline{j}$ let a clique tree tour on τ_i be given which satisfies $x_{c_0 4_i} + x_{4_i c_0} = 0$. Take out vertex c_0 from this tour, leaving a path.

Concatenate these paths together and link the ends to create the desired tour. None of the arcs which count were destroyed to create the desired tour. Moreover, $5 + |I| - 1 + 3|L|$ arcs which count have been added. Hence by the relationships of the right hand sides of these clique tree inequalities which was described before, the desired tour is a clique tree tour.

stage group 7: $|\overline{I}|$ stages

(ii)$x_{24_{\overline{j}}} = 0$

(iii) Reverse the previous tour.

stage group 8: $|\overline{I}|$ stages

(ii)$x_{4_{\overline{j}}1} = 0$

(iii) Using symmetry arguments, a tour like the tours for step (iii) of the two previous stages can be found.

stage group 9: $|\overline{I}|$ stages

(ii)$x_{14_{\overline{j}}} = 0$

(iii) Reverse the previous tour.

These last four stages are repeated for all $\overline{j} \in \overline{I}$

5.3.2 middle phase

stage group 10: 2 stages

(ii) $x_{as} = 0$ for $s = 5, d$

(iii) Let a clique tree tour on the medium clique tree that contains the arc c_0s be given. Replace arc c_0s with $c_0 \mathrel{-\!\!+\!\!\triangleright} 3 \mathrel{-\!\!+\!\!\triangleright} m \mathrel{-\!\!+\!\!\triangleright} 2 \mathrel{-\!\!+\!\!\triangleright} b$ zig-zag $1 \mathrel{-\!\!+\!\!\triangleright} a \rightarrow s$ to form the desired tour. Since none of the arcs which count were destroyed and $5+3|L|$ arcs which count have been added, the desired tour is a clique tree tour.

stage group 11: $n-8$ stages

(ii) $x_{at} = 0$ for $t \in V(BclG)\backslash\{5, d, e\}$

(iii) Consider a clique tree tour on the big clique tree that has the arc dt. Replace arc dt with $d \rightarrow b \mathrel{-\!\!+\!\!\triangleright} 2 \mathrel{-\!\!+\!\!\triangleright} m \mathrel{-\!\!+\!\!\triangleright} 1 \mathrel{-\!\!+\!\!\triangleright} a \rightarrow t$ to obtain the desired tour. Since 4 arcs which count have been added and none which count have been destroyed, the desired tour is a clique tree tour.

stage group 12: 2 stages

(ii) $x_{sb} = 0$ for $s = 6, e$

(iii) Replace arc sc_0 in a medium clique tree tour with $s \rightarrow b \mathrel{-\!\!+\!\!\triangleright} 2$ zig-zag $a \mathrel{-\!\!+\!\!\triangleright} 1 \mathrel{-\!\!+\!\!\triangleright} m \mathrel{-\!\!+\!\!\triangleright} 3 \mathrel{-\!\!+\!\!\triangleright} c_0$.

stage group 13: $n-8$ stages

(ii) $x_{tb} = 0$ for $t \in V(BclG)\backslash\{6, d, e\}$

(iii) Replace arc te in a big clique tree tour with $t \rightarrow b \mathrel{-\!\!+\!\!\triangleright} 2 \mathrel{-\!\!+\!\!\triangleright} m \mathrel{-\!\!+\!\!\triangleright} 1 \mathrel{-\!\!+\!\!\triangleright} a \rightarrow e$.

stage group 14: $n - 6 - |I| - |\overline{I}|$ stages

(ii) $x_{s2} = 0$ for $s \in V(SmclG)\backslash(\{c_0\} \cup \{4_{\overline{j}} \,|\, \overline{j} \in \overline{I}\})$

(iii) Replace arc $s3$ in a big clique tree tour with $s \rightarrow 2 \mathrel{-\!\!+\!\!\triangleright} b \rightarrow a \mathrel{-\!\!+\!\!\triangleright} 1 \mathrel{-\!\!+\!\!\triangleright} m \mathrel{-\!\!+\!\!\triangleright} 3$.

stage group 15: $n - 6 - |I| - |\overline{I}|$ stages

(ii)$x_{2s} = 0$ for $s \in V(SmclG)\backslash(\{c_0\} \cup \{4_{\overline{j}} \,|\, \overline{j} \in \overline{I}\})$

(iii) Reverse the previous tour.

Now, for reasons which will become clear near the end of the proof, it is important to have one vertex β such that arc 1β is allowed for the tours of step (iii) even though the arc $\beta 1$ isn't allowed. However, which vertex one should choose to be β turns out to depend on the structure of the big clique tree.

First suppose a clique tree tour for the big clique tree has at least three arcs which don't count. Then let a clique tree tour on the big clique tree where $x_{d3} = 1$ be given, and choose another arc $\gamma\beta$ which doesn't count in this clique tree tour.

Now suppose a clique tree tour for the big clique tree has only two arcs which don't count. Then define $\beta := e$

stage group 16: $n - 7 - |I| - |\overline{I}|$ stages

(ii)$x_{1s} = 0 \qquad$ for $s \in V(SmclG)\backslash(\{c_0\} \cup \{4_{\overline{j}} \,|\, \overline{j} \in \overline{I}\} \cup \{\beta\})$

(iii) Replace arc $3s$ in a big clique tree tour with $3 \rightarrowtriangle m \rightarrowtriangle 2 \rightarrowtriangle b \longrightarrow a \rightarrowtriangle 1 \longrightarrow s$.

stage group 17: $n - 6 - |I| - |\overline{I}|$ stages

(ii)$x_{s1} = 0 \qquad$ for $s \in V(SmclG)\backslash(\{c_0\} \cup \{4_{\overline{j}} \,|\, \overline{j} \in \overline{I}\})$

(iii) Reverse the previous tour. After this stage groups, the symmetry of the constraints has been broken.

stage 18

(ii) $x_{1a} + x_{a1} = 0$

(iii) Replace arc $3e$ in a big clique tree tour with $3 \rightarrowtriangle 1 \rightarrowtriangle m \rightarrowtriangle 2 \rightarrowtriangle b \longrightarrow\longrightarrow a \longrightarrow e$.

stage 19

(ii) $x_{ba} = 0$

(iii) Replace arc $d3$ in a big clique tree tour with $d \longrightarrow b \longrightarrow a \rightarrowtriangle 1 \rightarrowtriangle m \rightarrowtriangle 2 \rightarrowtriangle 3$.

stage 20

(ii) $x_{ab} = 0$

(iii) Replace arc $3d$ in a big clique tree tour with $3 \rightarrowtriangle 2 \rightarrowtriangle m \rightarrowtriangle 1 \rightarrowtriangle a \longrightarrow b \longrightarrow d$.

stage 21

(ii) $x_{2a} = 0$

(iii) Replace arc $d3$ in a big clique tree tour with $d \longrightarrow b \rightarrowtriangle 2 \longrightarrow a \rightarrowtriangle 1 \rightarrowtriangle m \rightarrowtriangle 3$.

stage 22

(ii)$x_{a2} = 0$

(iii) Reverse the previous tour.

stage 23

(ii) $x_{b1} = 0$

(iii) Replace arc $3e$ in a big clique tree tour with $3 \twoheadrightarrow m \twoheadrightarrow 2 \twoheadrightarrow b \rightarrow 1 \twoheadrightarrow a \rightarrow e$.

stage 24

(ii)$x_{1b} = 0$

(iii) Reverse the previous tour.

stage 25

(ii) $x_{31} = 0$

(iii) Replace arc de in a medium clique tree tour with $d \rightarrow b \twoheadrightarrow 2 \twoheadrightarrow m \twoheadrightarrow 3 \twoheadrightarrow 1 \twoheadrightarrow a \rightarrow e$ and insert zig-zag into another arc which doesn't count.

stage 26

(ii)$x_{13} = 0$

(iii) Reverse the previous tour.

5.3.3 ending phase

stage 27

(ii) $x_{23} = 0$

(iii) Choose a clique tree tour on the tiny clique tree that has vertex e such that $x_{6e} + x_{e6} = 0$. A maximal path of arcs which count that contains c_0 either starts or ends at c_0 If it ends at c_0 then reverse it. Then place this path right after e in the tour. If the maximal path of arcs which count that contains d starts at d then reverse it. Place node b after d and replace arc ec_0 with $e \rightarrow a \twoheadrightarrow 1 \twoheadrightarrow m \twoheadrightarrow 2 \twoheadrightarrow 3 \twoheadrightarrow c_i \twoheadrightarrow \twoheadrightarrow c_0$. Remove the link right after b to form a path. For the other tiny clique trees, pick a clique tree tour satisfying $x_{c_0 4_i} + x_{4_i c_0} = 0$. Remove c_0, forming a path. Concatenate all these paths and link the ends together. This yields the desired tour.

stage 28

(ii) $x_{b2} + x_{2b} = 1$

(iii) Choose a clique tree tour on the tiny clique tree $\tau_{\bar{j}}$ that has vertex e such that $x_{c_0 4_{\bar{j}}} + x_{4_{\bar{j}} c_0} = 0$. A maximal path of arcs which count that contains e either starts or ends at e. If it ends at e then reverse it. Then place this path right after c_0 in the tour. If the maximal path of arcs which count that contains d starts at d then reverse it. Place node b after d and replace arc c_0e with $c_0 \twoheadrightarrow \twoheadrightarrow c_i \twoheadrightarrow 3 \twoheadrightarrow 2 \twoheadrightarrow m \twoheadrightarrow 1 \twoheadrightarrow a \rightarrow e$. Remove the link right after b to form a path. For the other tiny clique trees, pick a clique tree tour satisfying $x_{c_0 4_i} + x_{4_i c_0} = 0$. Remove c_0, forming a path. Concatenate all these paths and link the ends together. This yields the desired tour.

stage 29

(ii) $x_{32} = 0$

(iii) Place $a \twoheadrightarrow 1 \twoheadrightarrow m \twoheadrightarrow 3 \twoheadrightarrow 2 \twoheadrightarrow b$ into an arc that doesn't count in the medium clique tree and place zig-zag into another arc that doesn't count.

stage m5

(ii) :$x_{m3} = 0$

(iii) :It is necessary to examine two cases here, namely that the big clique tree has at least three arcs which don't count, and that it has only two such arcs. This is necessary since if the big clique tree has only two arcs which don't count, then to find the desired tour for the next stage, it is important that the arc $1e$ be allowed, even though arc $e1$ isn't allowed. But β is presently defined to be the only node such that the arc 1β is allowed but the arc $\beta 1$ isn't allowed. Hence it is important in the latter case for $\beta := e$, which is in fact the definition of β here. For the former case, however, this would be imposing too great a restriction.

Suppose a clique tree tour on the big clique tree has at least three arcs which don't count. Using the tour on the big clique tree that defined β, replace arc $d3$ with $d \rightarrow b \twoheadrightarrow 2 \twoheadrightarrow m \twoheadrightarrow 3$ and replace arc $\gamma\beta$ with $\gamma \rightarrow a \twoheadrightarrow 1 \rightarrow \beta$

Now suppose a clique tree tour on the big clique tree has only two arcs which don't count. This means that the big clique tree is only a comb having three teeth, two of

which are $\{5, d\}$ and $\{6, e\}$ and possibly a node $\overline{m}$ that is in the handle but not in any teeth. Then the desired tour is $d \rightarrow b \twoheadrightarrow 2 \twoheadrightarrow m \twoheadrightarrow 3 \twoheadrightarrow c_i \twoheadrightarrow\twoheadrightarrow c_j \rightarrow a \twoheadrightarrow 1 \rightarrow e \twoheadrightarrow 6 \twoheadrightarrow \overline{m} \twoheadrightarrow 4 \twoheadrightarrow 5 \twoheadrightarrow$.

stage m6

(ii) :$x_{3m} = 0$

(iii) :Suppose that a clique tree tour on the big clique tree has at least three arcs which don't count. Again use the tour on the big clique tree that defined β. Consider the two cases where a third arc that doesn't count is between 3 and β or where it is between β and d. In either case call this arc $\epsilon\kappa$. In the first case take the path from b to ϵ in the desired tour for the previous stage and reverse it inside the tour. This yields the desired tour. In the second case take the path from κ to γ in the desired tour for the previous stage and reverse it inside the tour. This yields the desired tour.

Now suppose that a clique tree tour on the big clique tree has only two arcs which don't count. This means the big clique tree is as defined before. Then the desired tour is $3 \twoheadrightarrow m \twoheadrightarrow 2 \twoheadrightarrow b \rightarrow d \rightarrow a \twoheadrightarrow 1 \rightarrow e \twoheadrightarrow 6 \twoheadrightarrow \overline{m} \twoheadrightarrow 5 \twoheadrightarrow 4 \twoheadrightarrow c_i \twoheadrightarrow\twoheadrightarrow c_j \twoheadrightarrow$.

stage m7

(ii) :$x_{m1} = 0$

(iii) :Choose a big clique tree tour with $x_{de} = 1$. Replace arc de in this tour with $d \rightarrow b \twoheadrightarrow 2 \twoheadrightarrow m \twoheadrightarrow 1 \twoheadrightarrow a \rightarrow e$

The polytope that remains consists only of tours that include the path $a \twoheadrightarrow 1 \twoheadrightarrow m \twoheadrightarrow 2 \twoheadrightarrow b$. Hence these vertices can be contracted into a single node. Furthermore, after this contraction, all the tours of the polytope are clique tree tours for the big clique tree. But since the big clique tree defines a facet, the dimension of this polytope is $(n-4)^2 - 3(n-4)$. In all $8n - 27$ stages were used to obtain this polytope, so a lower bound for the dimension of the asymmetric traveling salesman polytope is determined to be $n^2 - 3n + 1$. Since this is its actual dimension, the large clique tree defines a facet as well.

Hence since the large clique tree and the underlying graph G were arbitrary, then for any underlying graph G, all clique trees in $C_{=k+1} \cap T \cap M$ define facets.

Part IV
cloning

The outline of this proof on page #9 summarizes the progress made in the goal of showing that all clique trees except for the comb on 6 vertices define facets of the ATSP polytope. Let the positive integer k be given. The first three parts of the proof establish that if all (simple) clique trees in C_k define facets, then all (simple) clique trees in $C_{=k+1} \cap T \cap M$ define facets.

To complete the proof, the concept of cloning, developed by Balas and Fischetti, is useful. Applying their paper to this particular case, define F to be the set of all clique tree inequalities for clique trees in $C_{=k+1}$, and F_n to be the subset of those inequalities where the underlying graph G has n vertices. Let $\alpha x \leq \alpha_0$ be a member of F_n. Let h, k be two distinct vertices of the underlying graph. Define $\overline{G} = (\overline{V}, \overline{A})$ to be the complete digraph induced by $\overline{V} := V \backslash \{h\}$. Then h and k are clones if:

(a)$\alpha_{ih} = \alpha_{ik}$ and $\alpha_{hi} = \alpha_{ki}$ for all $i \in V(G)$

(b)$\alpha_{hk} = \alpha_{kh} = \max\{\alpha_{ik} + \alpha_{kj} - \alpha_{ij} \mid i, j \in V(G) \backslash \{h, k\}, i \neq j\}$

(c)with $\overline{\alpha}$ being the restriction of α to $\overline{A}$ and $\alpha_0 := \alpha_0 - \alpha_k h$, the inequality $\overline{\alpha} x \leq \overline{\alpha}_0$ for the $n-1$ node ATSP is in F_{n-1}.

If there are no clone pairs in $\alpha x \leq \alpha_0$, then $\alpha x \leq \alpha_0$ is said to be primitive. An important result is that if all the primitive inequalities of F define regular facets, then all inequalities of F define regular facets, where a regular facet is one which isn't trivial and isn't equivalent to any two-city subtour elimination inequality.

In this case F is all the inequalities corresponding to clique trees in $C_{=k+1}$. So in order to show that all the inequalities of F all define facets, it is sufficient to show that

for a subset S of F that contains all the primitive inequalities of F, all the members of S are non-trivial reqular facets. Define S to be all the inequalities corresponding to clique trees in $C_{=k+1} \cap T \cap M$. These clique trees have $k+1 \geq 2$ handles, so it is clear they all define non-trivial regular facets.

To show that S contains all the primitive inequalities of F, it is sufficient to show that every inequality in $F \backslash S$ has a clone pair in it. Let an arbitrary clique tree in $C_{=k+1} \backslash C_{=k+1} \cap T \cap M$ be given. The inequality corresponding to this clique tree is thus in $F \backslash S$. Suppose this (simple) clique tree has a free tooth containing at least three vertices. Then one can choose distinct nodes h and k that are in this tooth, but are not in any handle. But then, h and k satisfy condition (a) for being a clone pair. Now, the coefficients in a clique tree inequality for a simple clique tree are all either 0 or 1. Also, the clique tree inequality involves summing up the weights of all the cliques in the clique tree. Thus, $\alpha_{ik} = \alpha_{kj} = 1$ implies that $\alpha_{ij} = 1$. Therefore, $\max\{\alpha_{ik} + \alpha_{kj} - \alpha_{ij} \mid i, j \in V(G) \backslash \{h, k\}, i \neq j\} \leq 1$. By choosing i to be another vertex in this tooth and j to be a vertex in another tooth, one sees that the above is actually an equality. So, since $\alpha_{hk} = \alpha_{kh} = 1$, condition (b) is satisfied as well. Condition (c) is satisfied too, so h and k are a clone pair.

Suppose this clique tree has a handle that has a middle having at least two vertices. Choose distinct nodes h and k in this middle. Then, h and k satisfy condition (a) for being a clone pair. As before, $\max\{\alpha_{ik} + \alpha_{kj} - \alpha_{ij} \mid i, j \in V(G) \backslash \{h, k\}, i \neq j\} \leq 1$. Picking i and j both in the handle makes this inequality tight. So, since $\alpha_{hk} = \alpha_{kh} = 1$, condition (b) is satisfied. Condition (c) is satisfied too, so h and k are a clone pair.

Therefore, every inequality in $F \backslash S$ has a clone pair in it. As a result, all (simple) clique trees in $C_{=k+1}$ define facets. Therefore, we have established the implication: Every clique tree inequality for clique trees in C_k defines a facet implies every clique tree inequality for clique trees in C_{k+1} defines a facet. But we know that all clique tree inequalities for clique trees in C_1 define facets. Hence it follows that all simple clique

trees except for the comb on 6 vertices define facets.

Now cloning will be used once more. Let F be the set of clique tree inequalities for all the clique trees (simple or not) having two or more handles. Let F_n be those inequalities where the underlying graph has n vertices. Let S be the set of clique tree inequalities for all the simple clique trees having two or more handles. Then we wish to show that S contains all the primitive inequalities of F by showing that any inequality in $F \backslash S$ has a clone pair in it. Since S clearly contains only non-trivial regular facets, this is sufficient to prove that all inequalities in F are facet- defining.

Let an inequality in $F \backslash S$ be given and choose distinct nodes h and k which are both in the intersection of a handle H and a tooth T. Then h and k satisfy condition (a) for being a clone pair. Consider $\alpha_{ik} + \alpha_{kj} - \alpha_{ij}$. If $i, j \in H \cap T$, this evaluates to 2+2-2=2. If $i \in H \cap T$, and $j \in T$, this becomes 2+1-1=2. If $i \in H$, and $j \in T$, this becomes 1+1-0=2. Evaluating all the other cases shows that $\max\{\alpha_{ik} + \alpha_{kj} - \alpha_{ij} \mid i, j \in V(G) \backslash \{h, k\}, i \neq j\} = 2$. But $\alpha_{hk} = \alpha_{kh} = 2$, so condition (b) is satisfied. Finally one can verify condition (c), which holds since when h is removed, the size of the handle H and the tooth T both go down by one, so the right hand side of the clique tree inequality goes down by 2 as it should.

It was already known that all one handle clique trees (combs) define facets except for the comb on a graph having 6 vertices. This proof has established that all other clique trees define facets for the ATSP as well.

Part V
conclusion

The dimension reduction technique applied to proving that clique trees are facets for the ATSP is slow and methodical, but the eventual success of the method seems to be almost inevitable if one is patient enough. Unknown to the author at the time that this

proof was developed, a proof that clique trees define facets for the ATSP was found by Fischetti four years ago. [6] However, this proof demonstrates the usefulness of the dimension reduction technique.

Given this nature of the technique, one possible area of future research is having a computer help in the proving process. The author has already developed a simple computer program for checking most of the mechanical aspects of this proof. It is hoped that in some cases, a computer program could construct a facet proof as well. The author is working on some ideas along these lines that he plans to publish in a future paper, and would appreciate any help in this regard.

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